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A one-layer model to predict the time development of static armour

Modèle mono-couche de prévision du développement d'un blindage statique

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ABSTRACT

The development of static armour has been studied extensively from a number of different points of view. Given the hydraulic conditions and the initial composition of the bed, many relations have been proposed which predict the armoured distribution which develops.

There are fewer relations which predict the development of the armour layer in time and the change in transport rate as the armour develops. An attempt is made in this paper to use a simple, one-layer model to predict the development of static armour and a comparison is presented with results obtained using different transport formulae for non-uniform sediments.

It is shown that a simple model based on the propagation of kinematic waves can predict the development of self-armoured beds by simulating the variations in sediment transport rate at the downstream end of a laboratory flume and in predicting the composition of the static armour that is produced.

RÉSUMÉ

Le développement d'un blindage statique a été intensivement étudié à différents points de vue. Les conditions hydrauliques et la composition initiale du lit étant donnés, de nombreuses relations ont été proposées, qui prévoient la distribution du blindage qui se développe. Il y a moins de relations qui prévoient le développement de la couche de blindage en fonction du temps et le changement du taux de transport pendant que le blindage se développe. On tente, dans cet article, d'employer un modèle simple à une couche pour prévoir le développement d'un blindage statique ; une comparaison est présentée avec les résultats obtenus par différentes formules de transport pour des sédiments non-uniformes. On montre qu'un modèle simple basé sur la propagation d'ondes cinématiques peut prévoir le développement des lits auto-blindés en simulant les variations du taux de transport de sédiment à l'extrémité aval d'un canal de laboratoire et en prévoyant la composition du blindage statique produit.

Keywords: Sediment; model; armouring; static armour; sediment transport.

1 Introduction

It has been widely reported that, when a flow is applied to a non-uniform sediment, selective transport can lead to the formation of a coarser surface layer, through a process known as armouring. This process has been observed both experimentally and in the field and depends upon the flow, as well as the composition of the sediment mixture and the transported material.

A state of equilibrium for the coarsened surface can result from the progressive decreasing of sediment transport (development of static armour), from the stability of the coarsest fraction and the mobility of the finest fractions (semi-static armour) or from the equalization in composition between the transported and the bed material (dynamic armour).

It has been observed that all size fractions of the original bed are present in an armoured surface. Many attempts have been made to predict armour formation and to reproduce experimental and field data. At different times a number of different aspects of the problem have been considered:

- sediment and hydraulic conditions under which armour develops;
- effects induced by the non uniformity of grains;
- the significance of the composition of the subsurface material;
- erosion associated with the development and temporal evolution of the surface;
- and the transport rate (Lamberti and Paris, 1992).

A number of procedures have been developed to predict the development of armour. A procedure is referred to as being a single-step method if it directly predicts the final equilibrium features from the initial conditions of the flow and sediment and is called a multi-step method if it follows the changes in the relevant variables as the armoured surface develops in time (Sutherland, 1987).

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Mathematical models belonging to this second group have become more and more complex in an attempt to improve the simulation of the observed exchanges of material between the flow and the bed and of the formation of the armoured surface. While one-layer models confine the description of the phenomenon to the surface layer, analysing the changes in this composition due to the interaction with the flow, multi-layer models (Ribbernik, 1987; Di Silvio, 1991) introduce sub-layers between the bottom surface and the lower, undisturbed material and rigorously define the interactions occurring between them.

The present paper discusses the simulation of the changes through time of the sediment transport rate and the final surface composition obtained experimentally for static armour conditions. Similar work has been carried out by Proffitt (1980) and Paris (1991), but the present work aims to show that a simple one-layer model can reproduce experimental results. By comparing the results obtained from the adoption of different transport theories for non-uniform sediments it was hoped that conclusions could be drawn about the predictions of such transport theories and the use of hiding functions.

1.1 *On the static armour formation*

It has been reported that in a reach with non-uniform bed material exposed to a constant flow rate with no sediment input and such that all sizes of sediment are in motion, erosion causes an initial decrease in the slope of the bed (rotational degradation) followed, because of the changed value of the applied shear stresses, by a selective erosion (parallel degradation) leading to the stabilization of the armour surface. In particular, downstream of a dam, the composition of the bed-surface gradually changes, as the finest fractions are eroded and not replaced by any upstream supply: the coarsened surface acts as a protective layer for the undisturbed, underlying material until an increased flow disturbs the overlying armoured layer.

Experimentally, static armour formation has been reproduced in flumes with a sediment-free input at constant discharge (Proffitt, 1980; Lamberti and Paris, 1992; Tait *et al.*, 1992). Analysing these results, it can be noticed that, for some runs, in a first short interval a great amount of material is removed at a nearly constant transport rate and that, successively, the transport rate gradually decreases. Comparing the behaviour of the same mixture in a number of runs of differing flow rates, it can be noticed that this first phase is present only for high values of shear stress. Low values of initial shear stress appear to be associated with an initially-increasing sediment transport rate. Proffitt (1980) observed that the initial phase is characterised by the formation of transverse coarse bars on the bed and that only afterwards does armouring of the bed take place with the tendency of the coarse particles to produce an imbricated bed.

He observed that the development of an armoured bed involves both the erosion of the finer particles and re-arrangement of the coarser fractions. He also remarked that, during the second phase, a quasi-linear relation exists in a log-log plot between sediment transport rate and time. For all his runs the composition of the transported material remained nearly constant, agreeing with Gessler's (1970) theoretical statements, but in contrast with Tait *et al.*'s (1992) experimental results.

1.2 *Brief review of existing one-layer models*

An important aspect of the analysis of the coarsening of the bed surface is the definition of the thickness of the armoured layer. At one time it was considered that changes in composition only occurred in the topmost layer of sediment and did not affect the underlying, undisturbed parent bed material.

Some researchers, however, have reported that changes in the bed composition are not only restricted to a layer that is only one grain thick, that is, to the material exposed to the flow, but that a finer sub-layer also forms under the coarser surface layer. To reproduce these features theoretically, multi-layer models have been proposed, but in spite of their accuracy in armour description, they need the specification of a larger number of parameters than the existing one-layer models. In this study attention has been restricted to one-layer models. Hirano (1971) seems to have presented the first one-layer model. He examined both degradation and deposition in a reach and assumed that the changes of the character of the material is restricted within a layer of variable thickness on the bed surface (exchange layer). He further assumed that, during erosion, material coming from the undisturbed underlying sediment is incorporated into the exchange layer. For the development of armouring, if the porosity of the bed is treated as a constant during degradation, that is $\lambda(x,t) = \lambda_0$, the general formulation of the continuity equation, referred to volumes and applied to a unit width reach, becomes:

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_0}\frac{\partial (q_s)_{vol}}{\partial x} \tag{1}$$

where z is the bed elevation, t the time, $(q_s)_{vol}$ the volume rate of sediment transport per unit width and x the distance in the direction of the flow. This is the sum of the continuity equations derived for each grain size class:

$$\frac{\partial m_i}{\partial t} = \frac{1}{a}(t_i - p_i)\frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a(1-\lambda_0)}\frac{\partial t_i}{\partial x} \tag{2}$$

where p_i, m_i, and t_i, are respectively the fraction of the i-th sediment class, with average representative size D_i, in the original material, in the exchange layer of thickness a and in movement.

If the porosity of the sediment is not considered, the continuity equation becomes:

$$\frac{\partial z}{\partial t} = -\frac{\partial (q_s)_{vol}}{\partial x} \tag{3}$$

which can be written, for each grain size class, as:

$$\frac{\partial m_i}{\partial t} = \frac{1}{a}(t_i - p_i)\frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a}\frac{\partial t_i}{\partial x} \tag{4}$$

These equations correspond to the equations:

$$\frac{\partial a m_i}{\partial t} + p_i\frac{\partial (z-a)}{\partial t} + \frac{\partial (q_s)_{vol} t_i}{\partial x} = 0, \tag{5}$$

in Lamberti and Paris (1992), with the exception that a is treated as a variable.

To solve this set of equations it is necessary to specify the thickness of the active layer a. The simplest assumption is to assure that a is constant. Hirano used a value of a close to the maximum grain size (D_{max}) of the original mixture. Other authors suggested values ranging from D_{50} to D_{max} (D_{nn} represents the dimension such that nn% in weight of the mixture is finer).

To fit experimental results, a particular solution of the general problem expressed by the above system of equations must be found through the introduction of appropriate initial and boundary conditions. Unfortunately, the method used by Hirano to express them is not clear.

Ashida and Michiue (1971), used as a control volume the entire flume, imposed an active layer of fixed thickness, and assumed the replacement of the eroded sediment by the same weight of material having the same composition as the active layer.

This approach led to the equation:

$$m_i(t + \Delta t) = (1 + c) \cdot m_i(t) - c \cdot e_i(t) \tag{6}$$

where Δt is the time step, c is the fraction of the active layer eroded at each time step and e_i the fraction of the i-th class in the eroded material. The size distribution of the eroded sediment was found by applying a transport rate equation to the surface distribution. The solution of the model was obtained by treating c as a constant. This implies that, as it is proportional to the product $(q_s)_{vol} \cdot \Delta t$, during the development of the armour later longer time steps were adopted, as the value of the transport rate decreased with time. The stability of the solution depends on the value of the constant c.

An approach similar to that of Ashida and Michiue was presented by Bayazit (1975).

Proffitt (1980), using the same control volume as that used by Ashida and Michiue, imposed a balance in weight between the eroded material and the sediment supplied from the parent bed, for an active layer of fixed thickness, and obtained the equation:

$$m_i(t + \Delta t) = m_i(t) - c \cdot e_i(t) + c \cdot p_i \tag{7}$$

The main difference between those two formulations is that in the latter case the eroded material is replaced by the same weight of material coming from the undisturbed, underlying bed, while Ashida and Michiue assumed that the replacement material had the same composition as that of the active layer.

2 Hiding functions

The simplest, naïve approach to calculating the sediment transport of a graded sediment is to utilise a formula conceived for uniform sediment.

In such an approach, the total transported material (q_s) is related to the sediment transport of each grain size class of the mixture, thought of as a uniform material (q_s), through the relation:

$$q_s = \sum_i q_{si} \cdot p_i \tag{8}$$

where p_i is the fraction of the i-th sediment class, with average representative size D_i, in the original material.

Comparison with experimental results shows that the behaviour of the coarsest and finest fractions is not exactly reproduced by such an approach. Results show that they have, respectively, greater and lower mobilities than predicted, both in terms of the threshold of motion and for transporting conditions. The complex interactions between the flow and the mixture, which result in this response, are still to be completely understood, but must include both static (e.g. sheltering of the finest fractions and protrusion of the coarsest ones) and dynamic (e.g. turbulence and collisions) aspects.

To overcome these difficulties analytical and empirical functions have been introduced. These are normally referred to as hiding functions and are used to modify the threshold conditions of motion and sediment transport formulae for a uniform bed material, adapting them to the case of graded sediments. A complete outline of the problems can be found in Sutherland (1991).

2.1 General description

Threshold of motion

Indicating with τ_c the mean applied shear stress at the threshold of motion, or critical shear stress, for a uniform material with grain size D and specific weight γ_s, it is well known that the corresponding entrainment parameter Θ_c ($\Theta_c = \tau_c/(\gamma_s - \gamma)D$, where γ is the specific weight of the water) is a function of the grain Reynolds number ($Re = u_* D/\nu$, where u_* is the shear velocity of the flow and ν is the kinematic viscosity of the water). Given a uniform material and flow parameters, the value of Θ_c can be obtained using the Shields' diagram, which relates Θ_c to Re, or from one of the existing sediment transport formulae for the condition of initiation of motion.

In a mixture, the value of the entrainment parameter at threshold conditions for each grain size fraction, $\Theta_{cmi} = \tau_{cmi}/(\gamma_s - \gamma)D_i$ – where τ_{cmi} is the critical shear stress of that fraction in the mixture – can be related to that of a uniform material of the same size, $\Theta_{ci} = \tau_{ci}/(\gamma_s - \gamma)D_i$, using a hiding function, ε_{ci}, defined as:

$$\Theta_{cmi} = \varepsilon_{ci} \cdot \Theta_{ci}. \tag{9}$$

Since the coarsest sediment fractions in a mixture are less difficult to move than the same size in a uniform sized sediment, it follows that for the coarsest fractions $\varepsilon_{ci} > 1$ and for the finest fractions $\varepsilon_{ci} > 1$. Between these two extremes there is a sediment size D_u, for which no correction is required, that is $\Theta_{cmu} = \Theta_{cu}$, $\varepsilon_{cu} = 1$. The subscript "u" is used to denote uniform behaviour. So, for $D_i > D_u$, $\Theta_{cmi} < \Theta_{ci}$, while for $D_i < D_u$ we have $\Theta_{cmi} > \Theta_{ci}$; that is the coarser fractions move for lower values of the applied shear stress than in a uniform bed, while finer fractions move at a higher applied shear stress.

In the most general way, the threshold conditions in a given mixture can be represented, in the Θ–Re plane, by a deformed Shields' curve, which intersects the Shields' curve for uniform sediment at the point $P \equiv (u_* D_u/\nu, \Theta_{cu}) \equiv (u_* D_u/\nu, \Theta_{cmu})$ as in Figure 1. The degree of deformation away from the Shields' curve depends upon the degree of non-uniformity of the sediment mixture. As the mixture becomes more uniform, the curve approaches more closely that for uniform sediment and also their

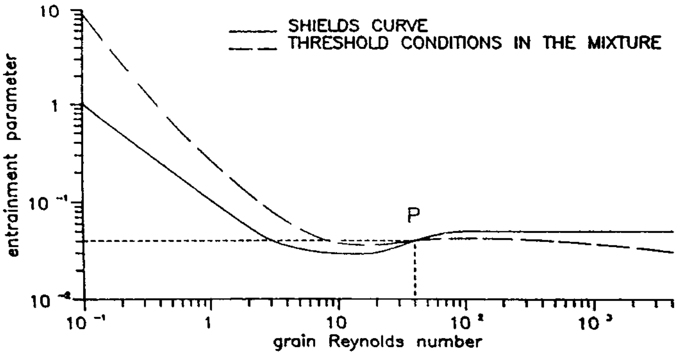

Figure 1 Shields curve for non-uniform sediments.

intersection point moves to the one representing the conditions of uniform sediment.

Referring to D_u, a second hiding function ε_i, can be defined:

$$\Theta_{cmi} = \varepsilon_i \cdot \Theta_{cu} \tag{10}$$

The two hiding functions are linked by the relation:

$$\varepsilon_i = \varepsilon_{ci} \cdot \frac{\Theta_{ci}}{\Theta_{cu}} \tag{11}$$

If we only consider sediments that are sufficiently large so that the entrainment parameter is constant with grain size, (Sutherland, 1991) then $\Theta_{ci} = \Theta_{cu} = \Theta_{cmu}$ for every size and Eq. (11) becomes:

$$\varepsilon_i = \varepsilon_{ci}; \tag{12}$$

in this case it can be stated that for $D_i > D_u$, $\Theta_{cmi} < \Theta_{cu}$ and for $D_i < D_u$, $\Theta_{cmi} > \Theta_{cu}$. This can be approximated by the relation:

$$\varepsilon_i = \left(\frac{D_i}{D_u}\right)^{-b} \tag{13}$$

(see Figure 2), with $b = 0$ when no hiding effects are present. In these circumstances:

$$\varepsilon_i = 1, \text{ as } \Theta_{cmi} = \Theta_{cu} = \Theta_{ci} \tag{14}$$

Equations (12)–(14) differ from those presented by Sutherland (1991) in which the assumption is made that the value of Θ_{ci} is the same for all fractions in the mixture. It can be verified that, when dealing with sediment mixtures with a variable uniform sediment threshold parameter, Eqs. (12)–(14) are no longer exact.

The concept of **equal mobility** of the grains is often discussed and in the context of threshold of motion this implies a situation in which all the fractions in a mixture possess the same critical shear stress:

$$\tau_{cmi} = \tau_{cu} = \text{Constant}. \tag{15}$$

This means that:

$$\Theta_{ci} = \frac{\tau_{cmi}}{(\gamma_s - \gamma)D_i} = \frac{D_u}{D_i} \cdot \frac{\tau_{cu}}{(\gamma_s - \gamma)D_u} = \frac{D_u}{D_i}\Theta_{cu}, \tag{16}$$

that is:

$$\varepsilon_i = \frac{D_u}{D_i} \tag{17}$$

and from Eq. (11):

$$\varepsilon_{ci} = \frac{\Theta_{cu}}{\Theta_{ci}} = \frac{D_u}{D_i} \cdot \frac{\Theta_{cu}}{\Theta_{ci}} = \frac{\tau_{cu}}{\tau_{ci}} \tag{18}$$

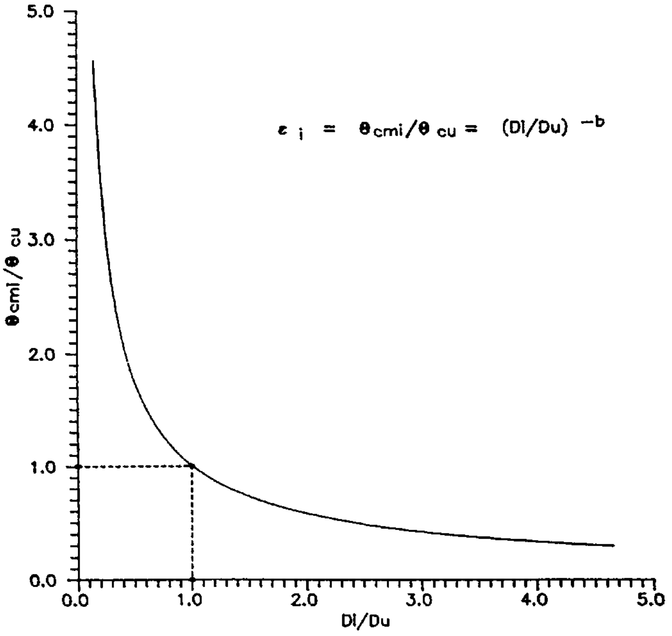

Figure 2 Hiding function ε_i as function of sediment size (Eq. (13)).

Sediment transporting conditions

Until now just the threshold of motion has been considered, but the interactions between the grains of the mixture are also important when sediment transport is taking place. Many relations have been proposed to evaluate the sediment transport rate for a flow transporting both bed load and suspended material. In the range of interest of the present studies on static armour formation, the total sediment transport can be identified with the bed load, whose features, such as saltation and rolling, are strictly linked with the composition of the bed surface.

A way to describe the interactions between sediment of different sizes is to use a hiding function (ξ_i) that introduces a modification to the applied shear stress for each grain size class (τ_i), compared with the mean shear stress on the bed (τ):

$$\tau_i = \xi_i \cdot \tau. \tag{19}$$

Depending upon the formula adopted, the effects of hiding for each grain size class can be represented by a modification to the applied shear stress, through ξ_i, or to the threshold value, through ε_i or ε_{ci}, in comparison with the uniform bed conditions.

As they have been conceived in different ways, generally speaking, there is no link between the types of hiding function, except for the relationship, expressed by Eq. (11), given above. As a result, comparisons are very difficult to make. For sediment transport formulae in which, for each grain size the effective applied shear stress and the critical shear stress only appear in the form τ_i/τ_{cmi} the correction that is introduced can be regarded either as a ξ_i, or a ε_{ci} type hiding function (Ribberink, 1987).

In the past, authors have proposed different forms of expressions for hiding functions, see Sutherland (1991) and Ribberink (1987), in an attempt to correct the results given by a specific sediment transport formula. It is important to notice that, when applying hiding functions derived for one sediment transport equation to a different one, for example, because of the lack of a specific one or because of the complexity of the appropriate formula, it should be remembered that some of them have been conceived in the context of a specific formula, for example Einstein's, and they are not adaptable to different sediment transport equations.

2.2 A brief review

A very basic review of some sediment transport formula and their hiding functions is now presented. Egiazaroff (1957) presented a sediment transport formula for uniform material in a polynomial form:

$$\frac{q_s'}{\gamma \cdot q} = K\left(\frac{\tau/(\gamma_s-\gamma)D}{\Theta_c} - 1\right) J^{0.5}, \tag{20}$$

where q_s' is the sediment transport rate by immersed weight, q is the flow rate and K is a coefficient, depending on the shape of the grains, whose value is 0.015 for spherical sand grains. For complete turbulence a value of $\Theta_c = 0.06$ is assumed. In 1965 he introduced a correction to this function, based on experimental data, to allow its use for a widely-graded sediment. Basically, for the case of complete turbulence, the entrainment parameter at threshold, equal for all sizes, $\Theta_{ci} = 0.06$, is substituted by the expression:

$$\Theta_{cmi} = \Theta_{ci}\left(\frac{\mathrm{Log}_{10}19}{\mathrm{Log}_{10}(19D_i/D_{avg})}\right)^2 \tag{21}$$

where the D_{avg} value is obtained by the combination of the average dimensions of the grain in movement and in the bed and is the size for which no correction is needed. The expression was proposed for D_i/D_{avg} ranging between 0.4 and 8.0. The ratio of the logarithmic terms can be seen as a hiding function $\varepsilon_i = \varepsilon_{is}$, depending on the sediment distribution through D_{avg}, and on the size being considered. This hiding function has been widely used, overcoming the problem in evaluating D_{avg} by using instead the values of D_g, the geometric mean diameter, or D_m, the mean diameter, or D_{50} of the bed distribution.

Ashida and Michiue (1971) applied this function to a formula of the Meyer-Peter Müller type, using a wider range of variation for D/D_{avg}. Calibration with experimental data suggested the correction:

$$\varepsilon_i = 0.85 \cdot \frac{D_{avg}}{D_i} \quad \text{if } \frac{D_i}{D_{avg}} < 0.4 \tag{22}$$

Day (1980), fitting experimental results, proposed a hiding function for the Ackers and White (1973) sediment transport formula. This formula expresses the sediment load as a function of the sediment mobility. Day modified the equation to account for non-uniform sediments by changing the threshold value of the sediment mobility. The hiding function used is of the ε_{ci} type and depends on the distribution of the mixture through a scaling size D_u, for which no correction needed. The hiding function depends on the characteristic diameters D_{16}, D_{50}, D_{84} and on the considered size.

On the basis of Proffitt (1980) experimental data, a hiding function was proposed by Proffitt and Sutherland (1983) for the Ackers and White formula in the ξ_i form, defined by Eq. (19). This formula allowed a correction to the mean applied shear stress for each grain size class; it depends on the size analysed and on a scaling size, which are functions both of the mixture and of the flow via the entrainment parameter $\Theta_{50} = \tau/(\gamma_s - \gamma)D_{50}$.

Even if beyond the purposes of this work, a brief mention of the work by Einstein (1950) is necessary. He first introduced the concept of hiding and translated it in his stochastic theory for uniform sediment into a ξ_i function, correcting the shear stress applied to the finer sizes of the mixture. No consideration was given to the impact on the coarser fractions.

To convert his transport formula to the graded case, he introduced also two more non-dimensional functions, X_E and Y_E. Unfortunately it is very difficult to compare his hiding function with other ones of the same type. In time, the fitting of new experimental data to the extended Einstein's formula has shown that the ξ_i functions give an excessive correction to the finest fractions. This lead Einstein and Chien (1953) and, later, Pemberton (1972) to propose a smoothed function ξ_i.

2.3 Adopted functions

In the present work both Meyer-Peter and Müller (1948) and Ackers and White (1973) sediment transport formulae are introduced in the model. First they are used in their original formulation, just correcting the results through the availability of sediment in the bed, but later hiding functions are adopted. The Egiazaroff (1965) function, as corrected by Ashida and Michiue (1971), is associated with the Meyer-Peter and Müller formula, with the reference size D_{avg} calculated as D_g, the geometric mean dimension of the active layer. The Ackers and White formula has been modified by using the Day (1980) hiding function.

3 Experimental data

Many reports of experiments on static armour formation can be found in the literature. Unfortunately few of these present detailed data on the time dependence of the main variables in the development of the armour layer and, in particular, on the values of total transported material and its distribution through time.

3.1 Proffitt (1980)

A tilting flume, with painted walls, 22 m long, and 0.605 m wide, was used. Recirculating clear water was supplied at a controlled temperature. The slopes used ranged between 0.0029 and 0.0043. Under nearly constant flow conditions, an armour coat was produced by parallel degradation in 18 runs for 4 different sediment mixtures which were very close to log-normal distribution. The sediment sizes ranged from a $D_{min} = 0.075$ mm to a $D_{max} = 38.10$ mm. The duration of the runs was between 24 and 98 hours. One of the runs was stopped at intervals to take photographs of the different phases of the armour formation. The author presented the plots reproducing the time dependence of the total transport rate and of the eroded material. Values of the initial and the armoured distributions are also available, as well as data on the hydraulic conditions.

3.2 Tait et al. (1992)

Data were obtained from 4 laboratory experiments on static armour formation carried out in a glass-sided, recirculating tilting flume which was 12.5 m long and 0.3 m wide. The slopes used ranged between 0.001 and 0.004. The hydraulic conditions were kept nearly constant and armour layers were produced by parallel degradation using 3 different sediment mixtures with sizes in the range of 0.05 and 7.0 mm. The durations of the runs were between 50 and 100 hours. The evolution of the armoured surface was followed by analysing photographs, taken every 6 hours, along the flume. Diagrams of the initial mixtures, the bed load and the armoured surface variations for some cases and evolution in time of sediment transport rate were presented. In Tait (1992) the changes in time of the bedload composition are reported for all the runs.

3.3 Notes on the sediment sampling and analysis

Attention must be paid in reading the experimental armour layer distribution curve, since it has been noticed that mistakes can be caused by confusing different procedures in sampling and analysis, see Proffitt (1980). Areal and volume sampling can be chosen to obtain a sample which is representative of the mixture and the obtained sample can be analysed both by area or by volume. The initial sediment composing the bed is generally sampled by bulk volume and analysed by weight. With the assumption of constant specific weight and porosity, this procedure is a classic one of volume by volume. While area by area and volume by volume analysis should give, for homogeneously distributed mixture, the same results, a relation is needed to pass from an area by volume, that is the case of an armour layer sampled areally and analysed by weight, to a volume by volume analysis. From the relation proposed by Kellerhals and Bray (1971) for an ideal mixture composed of cubes of 3 different sizes:

$$\mathrm{p(V-V)_i = Cp(A-V)_i D_i - 1} \tag{23}$$

where $\mathrm{p(V-V)_i}$ and $\mathrm{p(A-V)_i}$ are the fractions obtained volume by volume and area by area for the i-th sieve size, C is a constant equal to the total sample weight divided by the total adjusted weight and $\mathrm{D_i}$ is the i-th size, Proffitt (1980) derived the following equation for the natural mixtures that he used:

$$\mathrm{p(V-V)_i = Cp(A-V)_i D_i^{-\alpha}} \tag{24}$$

where α is in the range 0.4 to 0.54.

Such a conversion should be adopted to compare the results that a model gives for the armour layer distribution, generally presented as volume by volume, with the experimental data available, which is almost always, as area by volume.

4 Description of the model

The proposed model was developed from the scheme introduced by Hirano (1971), adapting it to match experimental conditions, that is, an initial bed consisting of known, non-uniform sediment and no sediment inflow at the upstream end of the flume. In the model different assumptions were tested to check the influence on the results of the characterisation of the exchange- or mixing-layer. The solution to the differential equations describing the model depends both on these assumptions and on the boundary and initial conditions chosen.

4.1 General formulation

As previously reported, Hirano (1971) first proposed a formulation of a one-layer model. For steady hydraulic conditions, the principle of continuity, applied to a control volume of width B and length dx, during an interval dt, provides the most general equation:

$$\frac{\partial z}{\partial t} = -\frac{1}{(1-\lambda_0)B}\frac{\partial (B(q_s)_{vol})}{\partial x} + \frac{a}{1-\lambda_0}\frac{\partial \lambda}{\partial t} + \frac{\lambda-\lambda_0}{1-\lambda_0}\frac{\partial a}{\partial t}, \tag{25}$$

where z is the bed elevation $(q_s)_{vol}$ the volume rate of sediment transport per unit width, λ the porosity of the bed, with initial value λ_0, is the thickness of the exchange layer, t time and x the distance in the direction flow.

For each grain size class of the mixture, the continuity equation takes the form:

$$\frac{\partial m_i}{\partial t} = \frac{(t_i - p_i)}{a}\frac{\partial z}{\partial t} - \frac{(q_s)_{vol}}{a(1-\lambda)}\frac{\partial t_i}{\partial x} + (m_i - p_i)\left(\frac{1}{(1-\lambda)}\frac{\partial \lambda}{\partial t} - \frac{1}{a}\frac{\partial a}{\partial t}\right) \tag{26}$$

for i = 1 to N, where N is the number of sediment classes. In Eq. (26) p_i, $m_i(x,t)$ and $t_i(x,t)$ are the fractions of the i-th sediment class, with average representative dimension D_i, in the original material, in the exchange layer and in movement, respectively. The unknown functions $m_i(x,t)$ and $t_i(x,t)$ satisfy the equations:

$$\sum_i m_i(x,t) = 1, \quad i = 1, N \tag{27a}$$

$$\sum_i t_i(x,t) = 1, \quad i = 1, N \tag{27b}$$

Equations (26) and (27) form a system of N + 2 independent equations. The system of equations contains 2N + 4 unknowns, $m_i(x,t)$ and $t_i(x,t)$ for i = 1 to N, a(x,t), $(q_s(x,t))_{vol}$, z(x,t) and $\lambda(x,t)$ and, so, to allow its solution, some further assumptions have to be made.

4.2 Assumptions and consequences

Through all the work three common assumptions are made.

Two of them are:

- the porosity of the bed, λ is a constant in time and space, that is, $\lambda(x,t) = \lambda_0$
- the control volume has a constant width, that is $B(x) =$ constant.

With these assumptions Eqs. (25) and (26) become:

$$\frac{\partial z}{\partial t} = -\frac{1}{1-\lambda_0}\frac{\partial ((q_s)_{vol})}{\partial x} \tag{28}$$

and

$$\frac{\partial(am_i)}{\partial t}+\frac{1}{1-\lambda_0}\frac{\partial(t_i(q_s)_{vol})}{\partial x}+p_i\frac{\partial(z-a)}{\partial t}=0,\ i=1\text{ to }N \quad (29)$$

Equation (29) corresponds to the previously presented Eq. (5).

The third assumption consists in expressing the transport rate of each sediment class, $t_i(q_s)_{vol}$, as the product of the value of the fraction in the mixing layer, m_i, and its volume transport rate capacity, $(q_{si})_{vol}$:

$$t_i(q_s)_{vol}=m_i(q_{si})_{vol}, \quad (30)$$

This leads to the equation:

$$\frac{\partial am_i}{\partial t}+\frac{1}{(1-\lambda_0)}\frac{\partial(m_i(q_{si})_{vol})}{\partial x}+p_i\frac{\partial(z-a)}{\partial t}=0. \quad (31)$$

The volume transport rate capacities, $(q_{si})_{vol}$, in the general case, are assumed to depend, through known hiding functions, on the composition of the mixing layer, that is, they are known functions of the variables $m_i(x,t)$. Equation (31) with the constraint:

$$\sum_i m_i(x,t)=1, \quad i=1,N \quad (27a)$$

or with the equation:

$$\frac{\partial z}{\partial t}=-\frac{1}{1-\lambda_0}\frac{\partial((q_s)_{vol})}{\partial x} \quad (28)$$

constitute the basis of the model. This system, however, is still indeterminate as it contains $N+2$ unknown variable, $m_i(x,t)$ for $i=1$ to N, a(x,t) and z(x,t), but only $N+1$ equations. To allow its solution, a further equation or assumption must be introduced.

The decision was made to analyse the behaviour of the model with three different assumptions to provide the extra constraint. These are referred to as SCHEME 1, SCHEME 2 and SCHEME 3.

SCHEME 1 is the simplest. It is obtained by imposing the following conditions along the flume and during the run:

- constant volume transport rate capacity for each sediment size-fraction, that is:

$$(q_{si})_{vol}(x,t)=(q_{si})_{vol} \quad (32)$$

- constant thickness of the mixing layer, that is:

$$a(x,t)=a_0 \quad (33)$$

Imposing these conditions Eq. (31) becomes:

$$a_0\frac{\partial m_i}{\partial t}+\frac{1}{1-\lambda_0}\frac{\partial(m_i(q_{si})_{vol})}{\partial x}+p_i\frac{\partial z}{\partial t}=0, \quad i=1,N \quad (34)$$

With the assumption that the third term is negligible in comparison with the second one, provided that this is different from zero, Eq. (34) for the movable and non-moveable sediment fractions, respectively, takes the form:

$$\frac{\partial m_i}{\partial t}+\frac{(q_{si})_{vol}}{a_0(1-\lambda_0)}\frac{\partial m_i}{\partial x}=0, \quad i=1,M, \quad (35a)$$

$$a_0\frac{\partial m_i}{\partial t}+p_i\frac{\partial z}{\partial t}=0, \quad i=M+1,N. \quad (35b)$$

It is assumed that only the size fractions 1 to M are in motion. The general solution to SCHEME 1 can be obtained from this system of equations together with the constraint:

$$\sum_i m_i(x,t)=1, \quad i=1,N \quad (27a)$$

SCHEME 2 is obtained by assuming that the thickness of the mixing layer is constant in time and space, that is, introducing the condition:

$$a(x,t)=a_0 \quad (33)$$

in Eq. (31).

The solution to the problem comes from a system of $N+1$ independent equations in $N+1$ unknown functions, that is $m_i(x,t)$, for $i=1$ to N, and z(x,t):

$$a_0\frac{\partial m_i}{\partial t}+\frac{1}{(1-\lambda_0)}\frac{\partial(m_i(q_{si})_{vol})}{\partial x}+p_i\frac{\partial z}{\partial t}=0, \quad i=1,M \quad (36a)$$

$$a_0\frac{\partial m_i}{\partial t}+p_i\frac{\partial z}{\partial t}=0, \quad i=M+1,N \quad (36b)$$

plus the constraint (27a): $\sum_i m_i(x,t)=1$.

The rearrangement of the equations produces an equivalent system consisting of Eq. (36) and an equation directly expressing the variation in time of the bed level:

$$\frac{\partial z}{\partial t}=-\frac{1}{(1-\lambda_0)}\sum_i\left(\frac{\partial(m_i)(q_{si})_{vol}}{\partial x}\right) \quad (37)$$

This scheme physically represents a mixing layer of constant thickness, moving downwards as erosion takes place, continuously being fed by the undisturbed, underlying parent bed. The value of the functions $m_i(x,t)$ refers to the ratio between the volume of each sediment fraction and the volume occupied by all the sediment fractions in the active layer. The last term, because of the assumptions of constant porosity and thickness of the active layer, does not vary in time and space.

In SCHEME 3 the term $(z(x,t)-a(x,t))$, representing the elevation of the bottom of the mixing layer, is treated as a constant. This assumption leads to the equations:

$$\frac{\partial am_i}{\partial t}+\frac{1}{(1-\lambda_0)}\frac{\partial(m_i(q_{si})_{vol})}{\partial x}=0, \quad i=1,M \quad (38a)$$

$$\frac{\partial am_i}{\partial t}=0, \quad 1=M+1,N, \quad (38b)$$

which have to be combined with the constraint:

$$\sum_i m_i(x,t)=1, \quad i=1,N \quad (27a)$$

From the rearrangement of these equations, the $(N+1)$-th equation can also take the form:

$$\frac{\partial a}{\partial t}=-\frac{1}{(1-\lambda_0)}\sum_i\left(\frac{\partial(m_i(q_{si})_{vol})}{\partial x}\right). \quad (39)$$

It can be noticed that Eqs. (37) and (39) obtained, respectively, by the summation of Eqs. (36) and (38), both represent the application of the principle of continuity to the mixture as a whole. In this scheme the derived functions $m_i(x,t)$ describe the variation of the ratio between the volume of each sediment fraction and the volume of all the sediment in an active layer of decreasing thickness a(x, t).

4.3 Analytical and numerical solutions

The particular solution to the differential equations describing the model in its three schemes, is dependent upon the initial and boundary conditions assumed, that is, on the values of the unknown variables:

$m_i(x, t)$ in SCHEME 1,
$m_i(x, t)$ and $a(x, t)$ in SCHEME 2,
$m_i(x, t)$ and $z(x, t)$ in SCHEME 3

for $t = 0$ along the flume (initial conditions) and for $x = 0$ during all the run (boundary conditions).

For all the schemes, the initial condition imposed on the sediment consisted of the equivalence of the composition of the parent bed, p_i, and of the mixing layer. Mathematically this is translated into the constraint:

$$m_i(x, o) = p_i \tag{40}$$

for $x = 0$ to LL, where LL is the length of the flume.

Depending on the further assumptions made, the differential equations characterising the schemes described above can be solved either analytically or numerically.

SCHEME 1

Equation (35) can be solved analytically. Referring to the movable sediment fractions, Eq. (35a) forms a set of linear differential equations of the first order and first degree that can be solved separately from the remaining equations. They represent M independent linear waves, describing the motion, in the positive x direction, of a perturbation to the initial conditions defined by the boundary conditions. The boundary conditions at the upstream end of the flume initiate the perturbations that move downstream with a celerity given by:

$$c_i = \frac{(q_{si})_{vol}}{a_0(1 - \lambda_0)}, \quad i = 1, M \tag{41}$$

The particular solution of the problem is obtained by adding to the initial conditions (40) the boundary conditions:

$$m_i(0, t) = 0, \quad i = 1, M, \tag{42a}$$

$$m_i(0, t) = p_i, \quad i = M + 1, N, \tag{42b}$$

referring to the moveable and non-moveable fractions respectively and simulating the absence of sediment inflow at the beginning of the flume during all the runs.

To match these conditions, the particular solution to Eq. (35a) becomes:

$$\begin{aligned} m_i(x, t) &= p_i \quad \text{if } t < \frac{x}{c_i} \\ m_i(x, t) &= 0 \quad \text{if } t > \frac{x}{c_i}, \qquad i = 1, M. \end{aligned} \tag{43a}$$

A fuller explanation of this result is given in HR Wallingford (1994). For the non-movable sediment fractions, that is, for $i = M + 1$ to N, the problem is simplified by observing that Eq. (35b), which states that the total amount of sediment along the flume does not vary in time, can be written as:

$$a_o m_i(x, t) + p_i z(x, t) = \text{Const.} = a_o p_i + p_i z(x, 0). \tag{35b}$$

Summing all these equations from $i = M + 1$ to N and substituting the obtained expression of $z(x, t)$ in each of them, the variation in time and space of the i-th fraction is a function of the initial distribution and of the actual distribution of the moveable fractions:

$$\begin{aligned} m_i(x, t) &= p_i \frac{\sum_{i=M+1,N} m_i(x, t)}{\sum_{i=M+1,N} p_i} \\ &= p_i \frac{1 - \sum_{i=1,M} m_i(x, t)}{\sum_{i=M+1,N} p_i}. \end{aligned} \tag{43b}$$

Physically, the relation (43a) indicates that at the end of the flume an initial interval of constant transport rate is followed, for $t > LL/(c_i)_{max}$, by a gradual decrease in sediment transport rate, After an interval of time equal to LL/c_i has elapsed, no further material of the i-th sediment fraction arrives at the end of the flume. From that time onwards, the value of $m_i(x, t)$ is equal to zero along the entire length of the flume.

As the celerity of the waves is directly proportional to the transport rate capacities of the sediment fractions, the transport of the finest particles will stop before that of the coarsest ones. When $t > LL/(c_i)_{min}$, the static armour condition is obtained.

SCHEME 2

The general formulation of the problem, represented by Eqs. (36) and (37), does not admit an analytical solution and a numerical approximation has been used. The numerical solution is obtained using a finite difference grid with a time step Δt and a space step Δx. Equation (36a), referring to the movable sediment fractions, that is, to $i = 1$ to M, with the use of Eq. (37), was approximated by:

$$\begin{aligned} &\frac{m_i(x, t + \Delta t) - m_i(x, t)}{\Delta t} a_0(1 - \lambda_0) \\ &= -\frac{(m_i(x, t)(q_{si})_{vol}(x, t) - m_i(x - \Delta x, t)(q_{si})_{vol}(x - \Delta x, t))}{\Delta x} \\ &+ p_i \frac{\sum_i (m_i(x, t)(q_{si})_{vol}(x, t) - m_i(x - \Delta x, t)(q_{si})_{vol}(x - \Delta x, t))}{\Delta x} \end{aligned} \tag{44a}$$

It can be observed that Eq. (36b), relative to the non movable fractions, coincides with Eq. (35b) of SCHEME 1. Therefore their solution, not requiring a finite difference approximation, is

$$\begin{aligned} m_i(x, t) &= p_i \frac{\sum_{i=M+1,N} m_i(x, t)}{\sum_{i=M+1,N} p_i} \\ &= p_i \frac{1 - \sum_{i=1,M} m_i(x, t)}{\sum_{i=M+1,N} p_i}. \end{aligned} \tag{44b}$$

The particular solution of the problems, as expressed by Eq. (44), is obtained by adding to the initial conditions (40) the relation:

$$z(x, o) = z_0 \tag{45}$$

and the boundary conditions:

$$m_i(0, t) = 0, \qquad i = 1, M, \tag{46a}$$

$$m_i(0, t) = \frac{p_i}{\sum_i p_i}, \quad i = M + 1, N, \tag{46b}$$

referring to movable and non-movable fractions, respectively. These equations simulate the absence of sediment inflow at the upstream end of the flume during all the runs.

In the particular case of assuming constant transport rate capacities of all the sediment fractions, the problem can be described by the equations:

$$a_0 \frac{\partial m_i}{\partial t} + \frac{(q_{si})_{vol}}{(1 - \lambda_0)} \frac{\partial (m_i)}{\partial x} + p_i \frac{\partial z}{\partial t} = 0, \quad i = 1, M, \tag{47a}$$

$$a_0 \frac{\partial m_i}{\partial t} + p_i \frac{\partial z}{\partial t} = 0, \quad i = M + 1, N, \tag{47b}$$

$$\frac{\partial z}{\partial t} = -\frac{1}{(1 - \lambda_0)} \sum_i \left((q_{si})_{vol} \frac{\partial (m_i)}{\partial x} \right). \tag{48}$$

This system was analytically solved by Gilbert (1992), as described in HR Wallingford (1994), with the initial conditions, Eq. (40) and the boundary conditions, Eq. (45).

As in SCHEME 1, the final condition of absence of transported material on an armoured surface is reached after the arrival at the end of the flume of the M-th perturbation wave, related to the M movable sediment fractions. In this case the calculation of the wave celerity requires some mathematical manipulation.

SCHEME 3

This scheme also does not allow an analytical solution. To perform a numerical approximation Eqs. (38a) and (39) are written introducing, again, a finite difference grid with time step Δt and space step Δx.

The equations become:

$$\frac{m_i(x, t + \Delta t) - m_i(x, t)}{\Delta t} a(x, t)(1 - \lambda_0)$$
$$= -\frac{(m_i(x, t)(q_{si})_{vol}(x, t) - m_i(x - \Delta x, t)(q_{si})_{vol}(x - \Delta x, t))}{\Delta x}$$
$$+ m_i(x, t) \frac{\sum_i (m_i(x, t)(q_{si})_{vol}(x, t) - m_i(x - \Delta x, t)(q_{si})_{vol}(x - \Delta x, t))}{\Delta x} \tag{48a}$$

and

$$\frac{a(x, t + \Delta t) - a(x, t)}{\Delta t}(1 - \lambda_0)$$
$$= \frac{\sum_i (m_i(x, t)(q_{si})_{vol}(x, t) - m_i(x - \Delta x, t)(q_{si})_{vol}(x - \Delta x, t))}{\Delta x} \tag{49}$$

with $i = 1$ to M.

Equation (38b), for the non-movable sediment fractions, can be directly solved as a function of the movable fractions using the equations:

$$m_i(x, t) = \frac{a(x, 0) p_i}{a(x, t)}, \quad i = M + 1, N \tag{48b}$$

The particular solution of the problem, as expressed by Eqs. (48) and (49), is obtained by adding to the initial conditions, Eq. (40), the relation:

$$a(x, 0) = a_0 \tag{50}$$

and the boundary conditions:

$$m_i(0, t) = f_i, \quad i = 1, N \tag{51}$$

where f_i is the distribution of the final armoured surface obtained experimentally or from the existing formulae, see, for example, Gessler (1971).

5 Test of the model

The model described above was tested using experimental data from Proffitt (1980) and Tait *et al.* (1992). Two aspects of the static armour formation were analysed: the variation in time of the total sediment transport rate and the final composition of the armoured surface.

5.1 Total transport rate in time

As a first step, the run 1–2 by Proffitt (1980) was simulated. This run was selected from the six runs that he performed with the same initial sediment (mixture 1). As the individual sediment transport measurements at the downstream end of the flume were not available, the curve fitted by Proffitt to the data was used. This curve consisted of two lines in a log-log plane: the first one reproduced the initial phase of constant transport rate, while the second part of the curve showed a decreasing sediment transport rate. Through all the present work a value of 30% was adopted for the porosity λ_0. This is a typical value for sediments of this nature. As the work concentrated on the differences between schemes, the precise value of the porosity was not considered important as it was constant for all the schemes.

SCHEME 1 – The value for the constant thickness of the mixing layer, a_0, was assumed equal to D_{100}, that is, equal to the maximum size of the parent bed material. The analytical solution, obtained by using Meyer-Peter and Müller (1948) and Ackers and White (1973) sediment transport formula, is presented in Figure 3. Even if the initial transport rate differs from the measured one, this simple scheme provides unexpectedly good agreement in the definition of the slope of the function describing the variation of the total sediment transport in time.

It was noticed that the slope of the solution depends not only on the sediment transport formula adopted, but also on the accuracy used to describe the mixture. The results presented in Figure 3 were obtained by schematising the mixture using the sieve data. A polynomial interpolation, based on splines, was used to increase the number of size classes in the model with

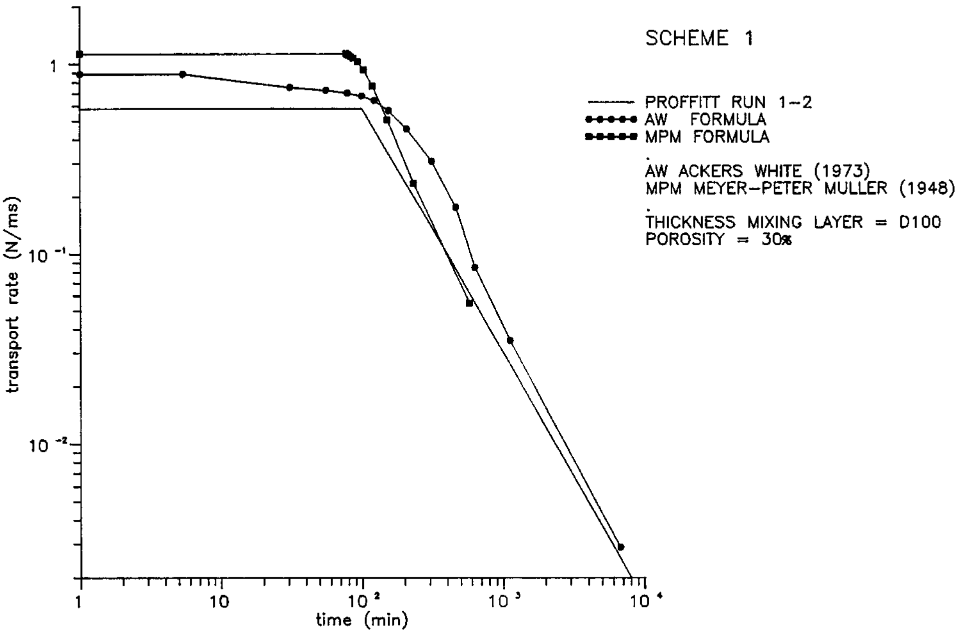

Figure 3 Scheme 1: Analyical solution, Meyer-Peter and Muller and Ackers and White sediment transport formulae.

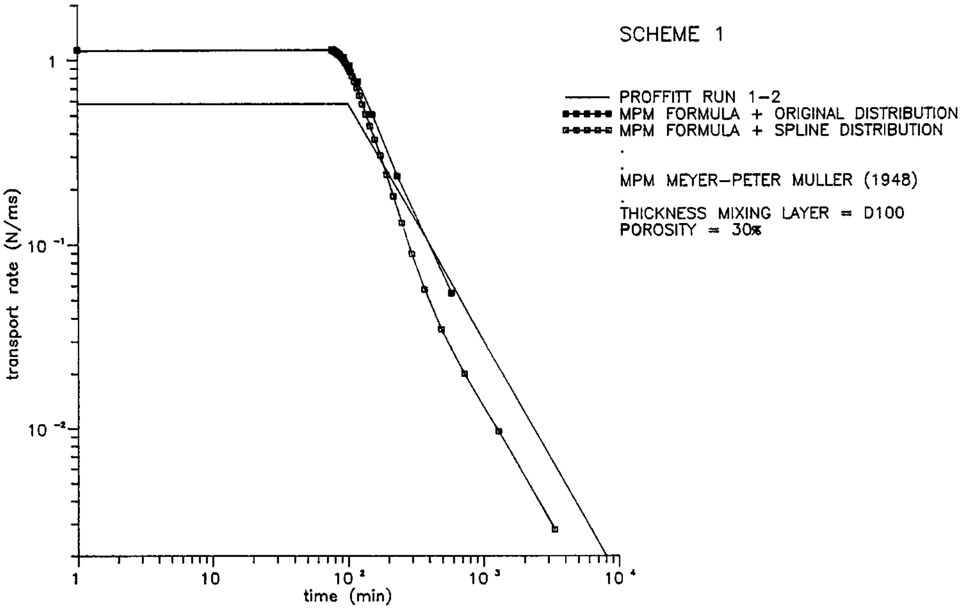

Figure 4 Scheme 1: Analytic solution, Meyer-Peter and Muller sediment transport formula, original and spline sediment distributions.

the aim of more closely approximating the continuous sediment distribution curve. More details on the spline theory can be found in Mizumura (1985). The results obtained using the Meyer-Peter and Müller formula with the sieving data and with the "spline distribution" are compared in Figure 4.

As a consequence of the linear dependence of each perturbation celerity, expressed by Eq. (41), on the product $a_0(1 - \lambda_0)$, variations in the porosity of the bed and in the thickness of the mixing layer produce a translation of the solution parallel to the time axis.

SCHEME 2 – For the mixing layer, a thickness equal to D_{100} was also assumed in this scheme. The bed of the channel did not exhibit any significant bed features and the experiments were purely degradational and so it was considered that the assumption was reasonable. The assumed thickness of the mixing layer is likely to affect the speed with which an equilibrium is achieved but is unlikely to affect the nature of the equilibrium. Initially no hiding factors were introduced. The analytical solution obtained by using Meyer-Peter and Müller and Ackers and White formulae is shown in Figure 5. By comparing the results of SCHEME 1 with SCHEME 2, it would appear that no advantage seems to be gained by the adoption of the more complex SCHEME 2.

No analytical solution is available for this scheme if hiding functions are introduced. In Figure 6 results are presented of the use of Meyer-Peter and Müller formula with Egiazaroff hiding function and Ashida and Michiue modification. The introduction of the hiding function produces, as expected, a lower value of the initial sediment transport rate, but no improvement in the simulation of the slope of the experimental data.

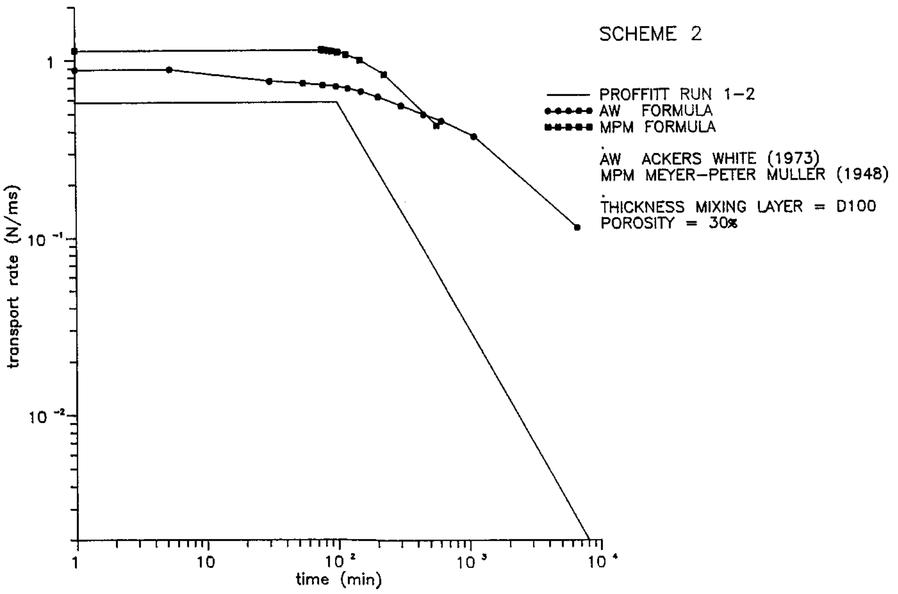

Figure 5 Scheme 2: Analytic solution, Meyer-Peter and Muller and Ackers and White sediment transport formulae.

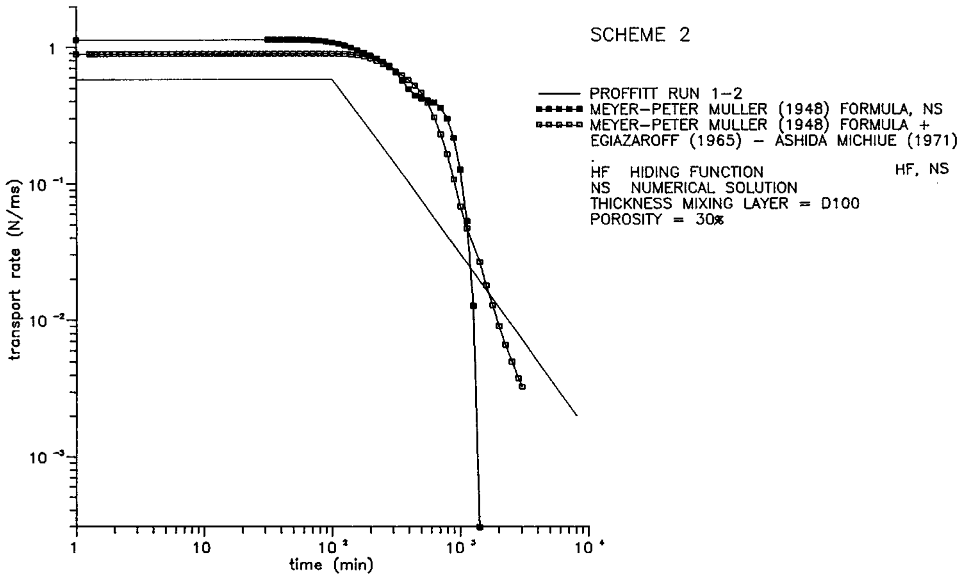

Figure 6 Scheme 2: Numerical solution, Meyer-Peter and Muller sediment transport formula with hiding functions.

SCHEME 3 – Initially the Meyer-Peter and Müller and Ackers and White formulas were applied, with no hiding correction, with the initial thickness of the mixing layer assumed to be equal to D_{100}. The numerical solution obtained with a ratio $\Delta x/\Delta t = 1$, and with the Meyer-Peter and Müller and Ackers and White formulas is presented in Figure 7. A very small improvement seems to come from the use of this scheme.

In order to consider the interactions between the fractions composing the sediment mixture, the Egiazaroff (1965) hiding function, corrected by Ashida and Michiue (1971), was coupled with the Meyer-Peter and Müller sediment transport formula and the Ackers and White sediment transport formula was associated with the Day (1980) correction. Results are presented in Figure 8. With reference to the Meyer-Peter and Müller formula, the slope of the solution obtained with hiding correction seems closer to the slope defined by Proffitt's data, but the solution underestimates the total eroded material. Uncertain results come from the Ackers and White formula and further investigations need to be carried out to understand the final trend of constant transport rate. It should be checked to see if this trend is reproducing a real behaviour of the model, depending on the formulation of the hiding function, or if it has just a numerical explanation.

The second step was to analyse Run 4 performed by Tait *et al.* (1992). The sediment transport formulae by Meyer-Peter and Müller and Ackers and White were introduced in SCHEME 1, adopting a mixing layer of fixed thickness equal to D_{100}. The results are presented in Figure 9. The Meyer-Peter and Müller formula gives a better fit to the experimental data, even if the initial value of the transport rate is overestimated. The Ackers and White sediment transport formula, on the contrary, underestimates the initial value of the transport rate. It can be seen that both of the solutions present a slope very close to the experimental one and it would be of interest to check if this occurrence is present in further applications of the scheme to different runs.

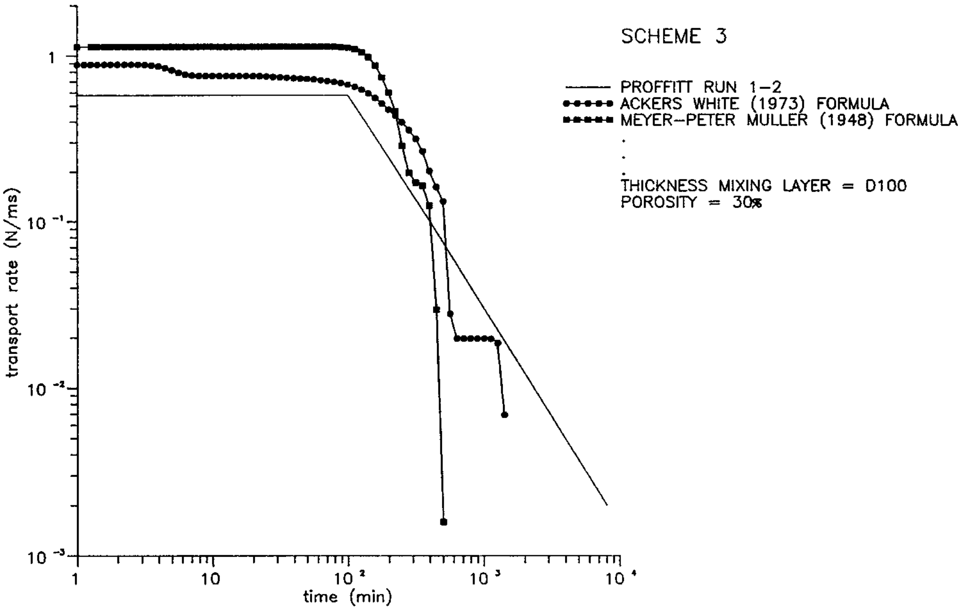

Figure 7 Scheme 3: Numerical solutions.

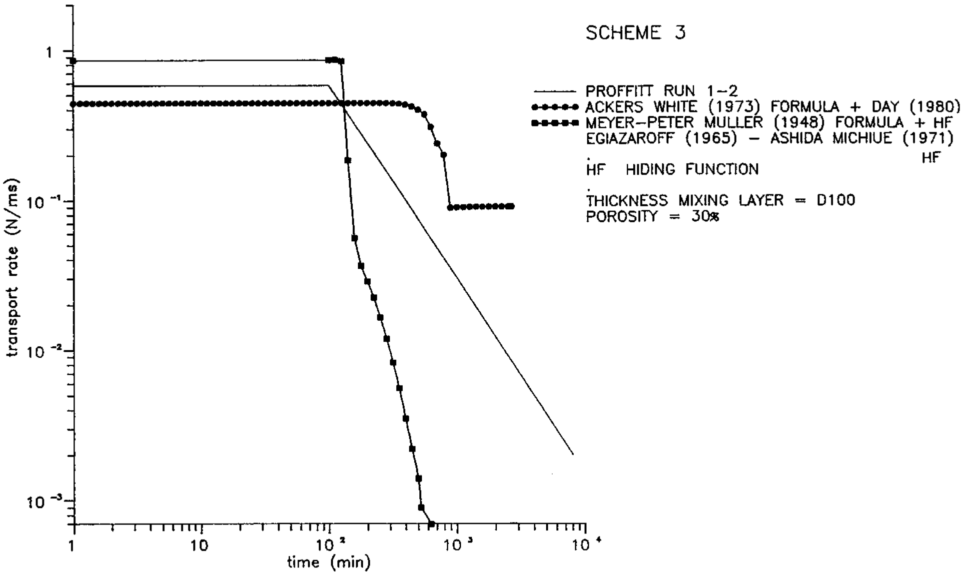

Figure 8 Scheme 3: Numerical solutions with hiding functions.

5.2 Sediment distribution at the surface

The composition of the armour layer obtained was compared with the ones produced using the different schemes of the model.

If no hiding functions are introduced, SCHEME 1, SCHEME 2 and SCHEME 3 give the same final distribution of the mixing layer. This layer is composed of those fractions of the mixture that would not be movable in a uniform sediment of their size. In the first two schemes the calculated percentages are referred to the fixed thickness of the mixing layer, while, in the third one, they are relative to the final value of it. Referring to Proffitt Run 1–2, Figure 10 shows the grading curves of the sampled armour layer and of the layers obtained by using both Meyer-Peter and Müller and Ackers and White sediment transport formula. Good agreement is obtained for the coarsest fractions with Meyer-Peter and Müller sediment transport formula. Thus the D_{50} and D_{95} values are reproduced to within approximately 10%. For the coarsest fractions the largest error occurs for approximately the D_{70} size where the error approaches 50%. For the sediment sizes smaller than D_{45} the errors increase rapidly so that for the D_5 size the error is approximately a factor of 30. The discrepancies relating to the finer fractions are due to the fact that the model, if no hiding functions are introduced, gives a final armour layer only composed of the non-moveable fractions.

Figure 11 shows the final distribution obtained by using Meyer-Peter and Müller sediment transport formula and Egiazaroff hiding function, modified by Ashida and Michiue. The agreement with the experimental data is good. As one would expect the inclusion of the hiding function has little impact on the coarser part of the grading curve but significantly changes the composition of the finer fractions. Thus the D_{50} and D_{95} values are reproduced to within approximately 10%. For the rest of the grading curve the predictions of the various sizes are within 50% of the observations, with many of the errors being smaller

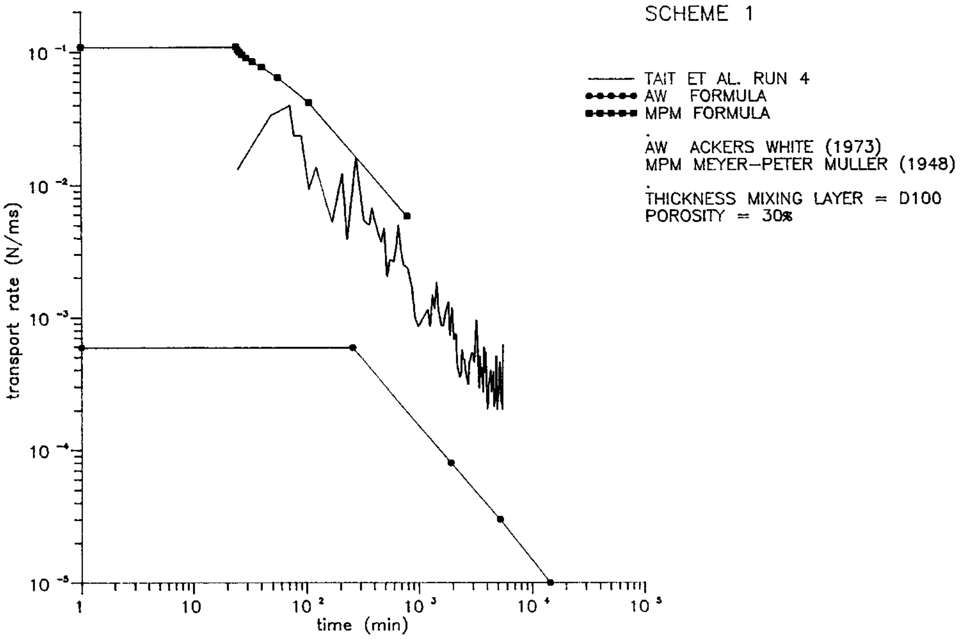

Figure 9 Scheme 1: Total sediment transport rate against time.

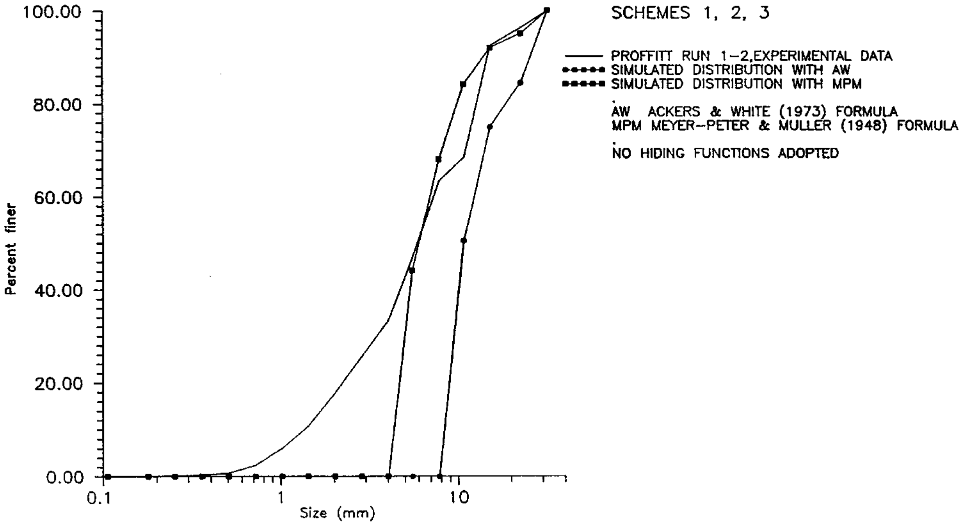

Figure 10 Predicted surface gradings with no hiding.

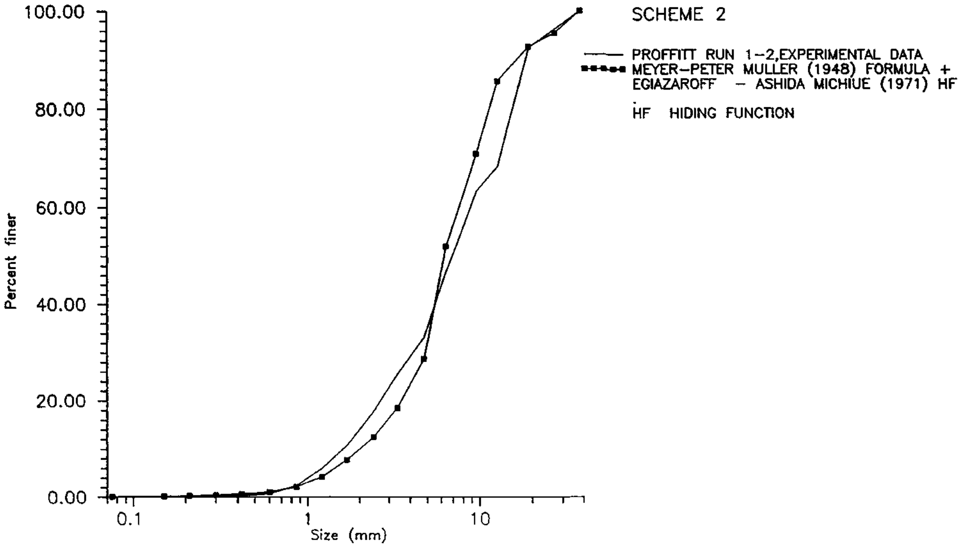

Figure 11 Predicted surface gradings with hiding functions.

than 50%. The largest errors occur for approximately the D_{70} and D_{20} sizes.

6 Implications for river mechanics

Though the experiments considered represent a very idealised system and though the models developed are simplistic in form, the results obtained have implications for river mechanics. The fact that a model has been developed that can provide a reasonable description of the system suggests that the development of armouring in a river can be at least approximately modelled by a similar approach using simple kinematic waves. The approach should be applicable not only to the prediction of static armour but also to the development of armour under mobile bed conditions.

A notable feature of the simulations of the laboratory experiments was the wide disparity of propagation speeds for the different sediment sizes. This was for a laboratory experiment with a relatively limited range of grain sizes. For natural sediments which commonly have a much wider range of sizes, the disparity in the propagation speeds will be even larger. This must be considered when interpreting observations within real river systems. An appreciation of both the propagation speeds and the time since an event occurred is required before we can correctly assess the downstream impact of events.

7 Conclusions and recommendations

7.1 Conclusions

A simple model based on the propagation of kinematic waves has been shown to predict the development of self-armoured beds. The model treats different sediment sizes independently and assumes that information on sediment supply propagates downstream at a rate dependent upon the sediment size.

It has been shown that for self-armouring experiments the model reproduces the characteristic shape of the plot of sediment transport rate with time. This curve shows an initially uniform transport rate followed, after a period of time, by a steadily reducing transport rate. The model indicates that this arises due to the different speeds with which information from the upstream boundary condition propagates down the flume.

A one-layer model was used to reproduce the variations of the sediment transport rate and the final composition of the surface, under static armour conditions. Three schemes were developed, based on different assumptions regarding the thickness of the mixing layer. The model was tested against experimental data.

SCHEME 1 gives the best results compared with transport rate data, despite its simplicity. In this scheme a spline function was introduced to shape the grading curve of the mixtures. Further investigations should be carried out to test the effectiveness of their presentation of the mixtures in both SCHEME 2 and SCHEME 3. The use of the theories of Meyer-Peter and Müller (1948) and Ackers and White (1973), Ashida and Michiue (1971) and Day (1980) in SCHEME 2 and SCHEME 3 did not produce the expected improvement in the fitting of experimental data. Different sediment transport formulae and hiding functions should be adopted to validate the obtained results. A good agreement in the final surface composition was obtained with SCHEME 2 and SCHEME 3.

Acknowledgements

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Notation

a = [L] thickness of the mixing layer
a_0 = [L] initial thickness of the mixing layer
B = [L] bed width
c = [-] coefficient in Eqs. (6) and (7)
c_i = [Lt^{-1}] celerity defined by Eq. (41)
$(c_i)_{min}$ = [LT^{-1}] minimum value of celerity defined by Eq. (41)
C = [-] constant in Eq. (23)
D = [L] characteristic dimension of a uniform sediment
D_{avg} = [L] average sediment size defined by Egiazarof (1965)
D_i = [L] average size of the i-the sediment class
D_g = [L] geometric mean sediment dimension
D_m = [L] mean sediment dimension
D_{max} = [L] maximum size of the initial sediment mixture
D_{min} = [L] minimum size of the initial sediment mixture
D_{nn} = [L] sediment size such that nn% in weight of the original mixture is finer, for example D_{35} and D_{50} are the sediment sizes for which, respectively, 35% and 50% of the sample is finer, by weight
D_u = [L] scaling dimension for hiding functions
e_i = [-] fraction of the i-the sediment class in the eroded material
f_i = [L] fraction of the i-the sediment class in the armoured surface
h = [L] flow depth
J = [-] slope
K = [-] Coefficient representing the shape of the grains
LL = [L] length of the flume
M = [-] number of the moveable fractions in the mixture
m_i = [-] fraction of the i-the sediment class in the exchange layer
N = [-] number of sediment fractions in the mixture
P = [-] Point on Shield's curve representing threshold conditions
p_i = [-] fraction of the i-the sediment class in the parent bed material
$p(A–V)$ = [-] fraction obtained with an area by area procedure
$P(V–V)$ = [-] fraction obtained with a volume by volume procedure
q = [M/T^3] discharge per unit width
q_s = [M/T^3] total sediment transport rate per unit width by dry weight

q_{si} = [M/T^3] sediment transport rate per unit width of the i-the sediment class, by dry weight
q'_s = [M/T^3] total sediment transport rate per unit width, by immersed weight
$(q_s)_{vol}$ = [L^{2}T^{-1}] sediment transport volume rate per unit width
Re = [-] grain Reynolds number; $Re = u_* D/\nu$
Rh = [L] hydraulic radius
t = [T] time
t_i = [-] fraction of the i-the sediment class in the transported material
U = [LT^{-1}] mean velocity of the flow
u_* = [LT^{-1}] shear velocity; $u_* = (\tau/\rho)^{0.5}$
x = [L] distance along the flow
X = [-] sediment concentration by dry weight $x = q'_s/\gamma_s$
X_E = [-] non-dimensional function defined by Einstein (1950)
Y_E = [-] non-dimensional function defined by Einstein (1950)
z = [L] bed elevation
z_0 = [L] initial bed elevation
α = [-] exponent in Eq. (24)
γ = [ML^{-2}T^{-2}] specific weight of water
γ_s = [ML^{-2}T^{-2}] specific weight of sediment
Δt = [T] time step
Δx = [L] space step
ε_{ci} = [-] hiding function defined by (9)
ε_i = [-] hiding function defined by (10)
Θ_c = [-] entrainment parameter for a uniform material of characteristic size D; $\Theta_c = \tau_c/(\gamma_s - \gamma)D$
Θ_{ci} = [-] entrainment parameter for a uniform material of characteristic size D_i; $\Theta_c = \tau_c/(\gamma_s - \gamma)D_i$
Θ_{cu} = [-] entrainment parameter for a uniform material of characteristic size D_u; $\Theta_{cu} = \tau_{cu}/(\gamma_s - \gamma)D_u$
Θ_{cmi} = [-] entrainment parameter in the sediment mixture for the i-th sediment class $\Theta_{cmi} = \tau_{cmi}/(\gamma_s - \gamma)D_i$
Θ_{cmu} = [-] entrainment parameter in the sediment mixture for the scaling size D_u; $\Theta_c = \tau_{cmu}/(\gamma_s - \gamma)D_u$
λ = [-] bed porosity
λ_0 = [-] initial bed porosity
ν = [L^2/T] kinematic viscosity of water
ξ_i = [-] hiding function defined by Eq. (19)
ρ = [M/L^3] density of the water
τ = [ML^{-1}T^{-2}] mean applied shear stress
τ_i = [ML^{-1}T^{-2}] corrected shear stress applied to the i-th sediment class of the mixture
τ_c = [ML^{-1}T^{-2}] critical shear stress for a uniform material of characteristic size D
τ_{ci} = [ML^{-1}T^{-2}] critical shear stress for a uniform material of characteristic size D_i
τ_{cu} = [ML^{-1}T^{-2}] critical shear stress for a uniform material of characteristic size D_u
τ_{cmi} = [ML^{-1}T^{-2}] critical shear stress in the mixture for the i-th sediment class
τ_{cmu} = [ML^{-1}T^{-2}] critical shear stress in the mixture for the scaling size D_u

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