Investigation of the compressibility of extra-high-velocity aerated flow
Investigation de la compressibilité du courant aéré à vitesses extrêmes

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ABSTRACT

Some fundamental topics of the compressibility influence of the extra-high-velocity aerated water flow in hydraulic engineering are studied in the present paper. A closed system of the simplified basic equations of the aerated water flow is deduced using the two-fluid model. A general formula of the sound velocity in the aerated water flow is derived using the sound analysis. It is found that the Wood adiabatic sound formula is a good approximation for calculating the sound velocity in the aerated water flow. Using the analysis of the order of the magnitude, it is found that the compressibility is important in the following cases. 1) The sound motion in the aerated water flow. 2) The aerated water flow with extra great depth. 3) The steady extra-high-velocity aerated water flow in which the mixture velocity is of the order of the sound velocity in it. It is shown that the adiabatic Mach number may act as the compressibility criterion in the steady aerated water flow. For the case of the steady homogeneous aerated water flow, a detailed compressibility standard is obtained, which is just the same as that in aerodynamics.

RÉSUMÉ

Le présent article étudie certains points fondamentaux de l'influence de compressibilité du courant aéré à vitesses extrêmes en génie hydraulique. Un système fermé des équations fondamentales simplifiées du courant aéré est dérivé d'un modèle à deux fluides. Une formule générale de la vitesse du son dans le courant aéré est dérivée d'une analyse sonique. Il est établi que la formule adiabatique de Wood fournit une bonne approximation de la vitesse du son dans le courant aéré. Une analyse dimensionnelle montre que la compressibilité est importante dans les cas suivante : 1) Mouvement sonique dans le courant aéré. 2) Courant aéré à très grande profondeur. 3) Courant aéré à vitesses extrêmes dans lequel la vitesse de mélange est de l'ordre de grandeur de la vitesse du son. Il est montré que le nombre adiabatique de Mach se peut servir de critère de compressibilité dans le courant aéré stable. Un critère de compressibilité est obtenu dans le cas d'un courant homogène aéré stable ; ce critère est identique à celui utilisé en aérodynamique.

Introduction

In the high velocity water flow with free surface, atmosphere air is usually entrained into and mixed with the water to create the aerated water flow, which is frequently observed in flows down steep chutes and spillways. Such a flow can also occur in a strongly inclined sewer pipe or industrial pipeline. It is believed that air entrainment is caused by the turbulent velocity fluctuations acting next to the air-water interface. Through this interface air is continuously being trapped and released, and the resulting air-water mixture may extend to the whole cross-section of the flow.

In the literature, the aerated water flow has been studied for the following reasons: (1) The entrained air increases the bulk of the flow which must be taken into account when designing spillway and chute sidewalls (Falvey 1980). (2) The presence of air within the boundary layer reduces the friction drag and the resulting increase of momentum must be considered when designing a ski jump and/or stilling basin downstream of a spillway (Chanson 1994). (3) The presence of air in the high velocity flow may be prevent or reduce the cavitation erosion damage (May 1987). (4) The presence of air enhances the air-water transfer of atmosphere gases, e.g. nitrogen, oxygen, carbon dioxide (Wilhelms & Gulliver 1989).

Furthermore, it has been found that the entrained air dramatically enhances the compressibility of the flow. The sound velocity in the aerated water flow is much lower than that in the flow of pure water or air at the same thermodynamic condition. The lowest sound velocity in the aerated water flow at ordinary condition is about 20 m/s (van Wijngaarden 1972, Zhao & Li 1996). On the other hand, along with the improvement of the technical competence for the dam construction and the advancement of the hydraulic engineering, more and more dams with more than 200-meter-height are constructed, and the dam height even exceeds 300 meters (Mermel 1991). This trend is more obvious in China. The resulting speed of the discharges from those dams may exceed 50 m/s, and the flows are bound to the aerated water flows. Then it can be inferred, according to the principle of aerodynamics, they must be considered as the compressible flows. Therefore, the compressibility of the extra-high-velocity aerated water flow in hydraulic engineering attracts more and more attentions (Cain & Wood 1977, Li 1993, Shuai 1995, Xia 1997, Yang 1998, Zhao 1998, Zhao & Li 1999).

The present paper deals with some fundamental topics of the compressibility of the extra-high-velocity aerated water flow in the hydraulic engineering. A closed system of the simplified basic equations of the aerated water flow is deduced. A general formula of the sound velocity in the aerated water flow is derived, and from a thorough discussion of this formula, a simplified formula, *i.e.* the Wood adiabatic formula, is proposed for

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calculating the sound velocity in the extra-high-velocity aerated water flow. The compressibility criterion of the aerated water flow is discussed. And a detailed compressibility standard of the steady homogeneous aerated water flow is obtained.

2 Fundamental Equations

The aerated water flow is generally a very complex air-water two-phase turbulent phenomenon. There exist several approaches for describing the two-phase flow. Each has its merits and demerits. The most reliable and frequently used one among them is the two-fluid model (Drew 1983, Liu 1993). Its fundamental hypothesis is that each phase of the two-phase mixture is a continuum in the local time-space range where it occupies. Based on this hypothesis the local instantaneous equations for the motion of each phase and the jump relations between the two phases in the interface are derived firstly. The macroscopically averaged equations of the two-phase flow are then deduced using some averaging method. However, the resulting macroscopically averaged equations are not closed and have very complex forms, and then they are not suitable for pragmatic purposes. In order to obtain a closed system of the simplified basic equations, it is important to construct a series of appropriate hypotheses and constitutive relations in accordance with the special characteristics of the considered flow.

The followings are our major assumptions for the extra-high-velocity aerated water flow in the hydraulic engineering:

1. No mass transfer occurs between the two phases throughout the flow.
2. The water-phase is incompressible and its temperature remains constant throughout the flow. Therefore, the density of the water-phase is constant.
3. The air-phase obeys the law of the ideal gas. Furthermore, it undergoes an adiabatic process throughout the flow.
4. The effects of the viscous stress and the turbulent fluctuations which include the “real” turbulent fluctuations and the fluctuations induced by the random distribution of the two phases are negligible.
5. There exists a local hydrostatic equilibrium throughout the flow. This means that the averaged pressures in each phase are equal at the same point in the flow field.
6. The drag and the added mass force mainly undertake the interfacial momentum transfer. Effects of other forces between the two phases are negligible.

The detailed analyses about these assumptions can be found in Zhao (1998). Here only a brief interpretation about the assumption of the adiabatic process of the air-phase is given.

The heat transfer occurred in one phase or between the two phases needs certain duration of time to reach its equilibrium. The characteristic time of this process in the air-water two-phase flow can be expressed by the internal and external temperature relaxation time of the air bubble, *i.e.*

$$\tau_{in} = \frac{\rho_1 c_{p1} d_0^2}{4\pi^2 k_1} \qquad \text{and} \qquad \tau_{ex} = \frac{\rho_1 c_{p1} d_0^2}{12 k_2}.$$

On the other hand, the change of the state of the air-phase in the flow is often caused by the change of velocity. The characteristic time of this process can be expressed as $\tau_u = d_0/u_{m0}$. For the case of the extra-high-velocity aerated water flow in the hydraulic engineering, $d_0 \sim 10^{-2}$m and $u_{\mathrm{m}0} \sim 10$m/s, which make $\tau_{in}/\tau_u \sim 10^2 \gg 1$ and $\tau_{ex}/\tau_u \sim 10 \gg 1$. Then the heat produced in the air-phase has no time to be conducted away, and its thermodynamic process is essentially adiabatic.

With these assumptions mentioned above, the resulting closed system of the simplified macroscopically averaged equations of the extra-high-velocity compressible aerated water flow in the hydraulic engineering can be written as:

$$\frac{\partial}{\partial t}(\alpha_k \rho_k) + \nabla \cdot (\alpha_k \rho_k \vec{u}_k) = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}(\alpha_k \rho_k \vec{u}_k) + \nabla \cdot (\alpha_k \rho_k \vec{u}_k \vec{u}_k) = -\alpha_k \nabla p + \alpha_k \rho_k \vec{g}_k + \vec{F}_k \tag{2.2}$$

$$\frac{d_1 s_1}{dt} = \frac{R \vec{F}_1 \cdot \vec{u}_r}{\alpha p} \tag{2.3}$$

$$p = \rho_1 R T_1 \tag{2.4}$$

$$\begin{aligned} \vec{F}_1 = -\vec{F}_2 = {} & C_D \alpha (1-\alpha)^3 \rho_2 |\vec{u}_r| \vec{u}_r - \\ & C_{VM} \alpha (1-\alpha) \rho_2 \left(\frac{d_2 \vec{u}_1}{dt} - \frac{d_1 \vec{u}_2}{dt} \right) \end{aligned} \tag{2.5}$$

where $\alpha_1 = 1 - \alpha_2 = \alpha$, $\vec{u}_r = \vec{u}_2 - \vec{u}_1$, and $s_1 = s_{10} + c_{vl} \ln[(p/p_{10})/(\rho_1/\rho_{10})^\gamma]$ · $(d_k/dt) = (\partial/\partial\bar{t}) + \vec{u}_k \cdot \nabla$ denotes the substantial derivative with respect to the *k*-phase. The subscript $k = 1,2$ denotes the air- and water-phase.

Eq. (2.5) is the constitutive relation of the drag force (the first term) and the added mass force (the second term) between the two phases. Here the relationship of the drag force proposed by Hench & Johnston (1972) and the Drew’s model of the added mass force (Drew 1983) are used. The parameter λ in the Drew’s model is set equal to 1 for the sake of the symmetry. According to Hench & Johnston (1972), $C_D = 41.2$ m^{-1}, which has the same dimension as the reciprocal of the length. In general, the added mass coefficient C_{VM} is a dimensionless variable. For the case of an isolated spherical bubble with a constant radius in a boundless flow field, the theoretical value of C_{VM} is 1/2. There is, however, some experimental evidence (Bataille et al 1990) shown that the value of C_{VM} may be much greater than 1/2 for a bigger bubble. It is also believed that C_{VM} is a function of the air concentration and other factors. Based on some experimental results (Eddington 1970, Salih 1980, Cain & Wood

1981, Bataille et al 1990), it can be deduced that $C_{VM} \sim 1$ and the relative velocity-slip between the two phases $\delta = \left|\vec{u}_r / \vec{u}_2\right| \sim 10^{-1}$ in the extra-high-velocity aerated water flow. The more detailed discussion can be found in Zhao (1998).

Eq. (2.1 – 5) constitutes a complete description of the extra-high-velocity compressible aerated water flow. With appropriate initial and boundary conditions, they can be solved, in principle, for the six unknowns α, ρ_1, $\vec{u}_1$, $\vec{u}_2$, p and T_1, as functions of the time and space.

3 Sound Velocity

Supposing that there is an infinitesimal perturbation (*i.e.* sound wave) propagating in the aerated water flow. For the sake of simplicity, we also assume that the mixture is motionless in its equilibrium state. For the perturbations of the flow quantities, which are denoted by an apostrophe here, are supposed to be small, Eq. (2.1 – 5) can be linearized. The resulting linearized equations are as follows:

$$\rho_1 \frac{\partial \alpha'}{\partial t} + \alpha \frac{\partial \rho_1'}{\partial t} + \alpha \rho_1 \nabla \cdot \vec{u}_1' = 0 \tag{3.1}$$

$$\frac{\partial \alpha'}{\partial t} - (1-\alpha) \nabla \cdot \vec{u}_2' = 0 \tag{3.2}$$

$$\begin{aligned}&[\rho_1 + C_{VM}(1-\alpha)\rho_2] \frac{\partial \vec{u}_1'}{\partial t} - \\ &\qquad C_{VM}(1-\alpha)\rho_2 \frac{\partial \vec{u}_2'}{\partial t} = -\nabla p'\end{aligned} \tag{3.3}$$

$$(1 + C_{VM}\alpha)\rho_2 \frac{\partial \vec{u}_2'}{\partial t} - C_{VM}\alpha\rho_2 \frac{\partial \vec{u}_1'}{\partial t} = -\nabla p' \tag{3.4}$$

$$p'/\rho_1' = \gamma p/\rho_1 = \gamma R T_1 = a_1^2 \tag{3.5}$$

Eq. (3.3 – 4) gives

$$\frac{\partial \vec{u}_2'}{\partial t} = \frac{\rho_1/\rho_2 + C_{VM}}{1 + C_{VM}} \frac{\partial \vec{u}_1'}{\partial t} \tag{3.6}$$

and

$$\frac{\rho(1 + C_{VM}\alpha) + \rho_2 C_{VM}(1-\alpha)}{1 + C_{VM}} \frac{\partial u_1'}{\partial t} + \nabla p' = 0 \tag{3.7}$$

Setting $\alpha' = H\rho_1'$ and substituting it into Eq. (3.1), we can obtain

$$(\alpha + \rho_1 H) \frac{\partial \rho_1'}{\partial t} + \alpha \rho_1 \nabla \cdot \vec{u}_1' = 0 \tag{3.8}$$

Elimination of $\vec{u}_1$ from Eq. (3.7 – 8), in combination with Eq. (3.5), gives

$$\frac{\partial^2 \rho_1'}{\partial t^2} = \frac{(1 + C_{VM})\alpha\rho_1 a_1^2}{(\alpha + \rho_1 H)[\rho_1(1 + C_{VM}\alpha) + \rho_2 C_{VM}(1-\alpha)]} \nabla^2 \rho_1' \tag{3.9}$$

This is a typical equation of wave motion and the wave velocity is given by

$$a_m = a_1 \sqrt{\frac{(1 + C_{VM})\alpha\rho_1}{(\alpha + \rho_1 H)[\rho_1(1 + C_{VM}\alpha) + \rho_2 C_{VM}(1-\alpha)]}} \tag{3.10}$$

On the other hand, it is well known that sound velocity can be expressed as

$$a = \sqrt{dp/d\rho} \tag{3.11}$$

In the present case, the mixture density is defined as

$$\rho_m = \alpha\rho_1 + (1-\alpha)\rho_2 \tag{3.12}$$

Eq. (3.11) can then be rewritten as $a_m = \sqrt{p'/\rho_m'}$, where $\rho_m' = (\rho_1 - \rho_2)\alpha' + \alpha\rho_1'$. Combining with Eq. (3.10), we can obtain

$$H = -\frac{\alpha(1-\alpha)(\rho_1 + C_{VM}\rho_2)}{\rho_1[(1-\alpha)\rho_1 + (\alpha + C_{VM})\rho_2]} \tag{3.13}$$

and

$$a_m = a_1 \sqrt{\frac{\rho_1[(1-\alpha)\rho_1 + (\alpha + C_{VM})\rho_2]}{\alpha\rho_2[\rho_1(1 + C_{VM}\alpha) + \rho_2 C_{VM}(1-\alpha)]}} \tag{3.14}$$

Eq. (3.14) is the general formula of sound velocity in aerated water flow. There are some special cases of Eq. (3.14), which will be discussed firstly.

1. If $C_{VM} = 0$, Eq. (3.14) will become

   $$a_m = a_1 \sqrt{1 + \frac{1-\alpha}{\alpha}\frac{\rho_1}{\rho_2}} \tag{3.15}$$

   which is identical with the so-called stratified flow sound velocity reported as Eq. (28 – 29) by Trapp & Ransom (1982) if the effects of the interfacial heat transfer and the compressibility of the water-phase are negligible. It is also very close to that reported as Eq. (12) by Nguyen et al (1981).
2. If $C_{VM} \to \infty$, Eq. (3.6) will become $(\partial \vec{u}_2')/(\partial t) = (\partial \vec{u}_1')/(\partial t)$. This means that there is no velocity-slip between the two phases and the mixture can be considered homogenous. In this case, Eq. (3.14) returns to the Wood adiabatic formula, *i.e.*

   $$a_m = a_1 \sqrt{\rho_1/(\alpha\rho_m)} = \sqrt{\frac{\gamma p}{\alpha\rho_m}} \tag{3.16}$$

3. If the effect of the mass of the air-phase is negligible, Eq. (3.14) will become

$$a_m = \sqrt{\frac{(1+\alpha/C_{VM})\gamma p}{\alpha(1-\alpha)\rho_2}} \tag{3.17}$$

which is identical with the formulae reported as Eq. (37b) by Crespo (1969) and Eq. (3.21) by Caflisch et al (1985). However, the limitation for α is more lenient in the present paper.

The predictions of Eq. (3.14) are plotted in figure 1. It is shown that very weak interaction between the two phases, *i.e.* very small value of C_{VM}, can cause great reduction of the sound velocity in the aerated water flow. If the interaction has got some certain intensity (e.g. the magnitude of C_{VM} is 1 or more), no essential difference exists between the predictions of Eq. (3.14) and that of the Wood adiabatic formula Eq. (3.16). This is just the case of the aerated water flow in the hydraulic engineering. The experimental results of the sound velocity in the steady area of self-aerated water flow in a steep chute (Shuai 1995) are also plotted in figure 1, which is very close to the prediction of the Wood adiabatic formula. Therefore, it is reasonable using the Wood adiabatic formula to calculate the sound velocity in the aerated water flow in the hydraulic engineering.

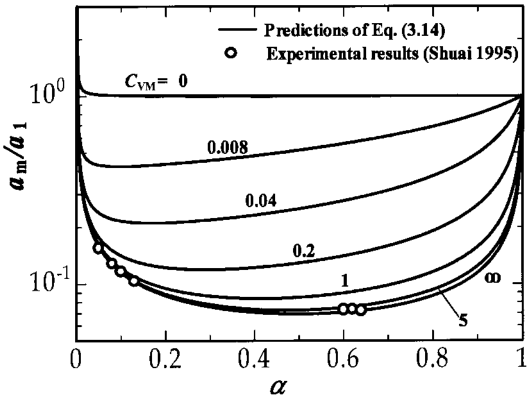

Fig. 1. Sound velocity in the aerated water flow.

4 Compressibility Criterion

The compressibility of the aerated water flow is a characteristic of the whole air-water mixture. It is then necessary to define the mixture velocity

$$\vec{u}_m = \frac{1}{\rho_m}[\alpha\rho_1\vec{u}_1 + (1-\alpha)\rho_2\vec{u}_2] \tag{4.1}$$

as well as the mixture density which is defined by Eq. (3.12). Their control equations can be deduced by taking the sum of Eq. (2.1) and (2.2) over k = 1,2, respectively. The resulting equations read as follows:

$$\frac{\partial\rho_m}{\partial t} + \nabla\cdot(\rho_m\vec{u}_m) = 0 \tag{4.2}$$

$$\frac{\partial}{\partial t}(\rho_m\vec{u}_m) + \nabla\cdot(\rho_m\vec{u}_m\vec{u}_m) = -\nabla p + \rho_m\vec{g} - \nabla\cdot\left[\frac{\alpha(1-\alpha)\rho_1\rho_2}{\rho_m}\vec{u}_r\vec{u}_r\right] \tag{4.3}$$

For the present case of aerated water flow in the hydraulic engineering, $\rho_1/\rho_2 \sim 10^{-3}$ and $\vec{u}_1 \sim \vec{u}_2$, then $\rho_m \approx (1-\alpha)\rho_2$ and $\vec{u}_m \approx \vec{u}_2$ unless α is very close to 1. Then the last term of Eq. (4.3) becomes approximately $-\nabla\cdot(\alpha\rho_1\vec{u}_r\vec{u}_r)$, and Eq. (4.2) can yield approximately

$$\frac{d_m\alpha}{dt} = (1-\alpha)\nabla\cdot\vec{u}_m \tag{4.4}$$

Using the definition of s_1, the pressure p can be expressed as

$$p = p(\rho_1, s_1) = p_0\left(\frac{\rho_1}{\rho_{10}}\right)^{\gamma}\exp\left(\frac{s_1 - s_{10}}{c_{v1}}\right),$$

which yields

$$\left(\frac{\partial p}{\partial\rho_1}\right)_{s_1} = \frac{\gamma p}{\rho_1} = \gamma RT_1 = a_1^2 \tag{4.5}$$

$$\left(\frac{\partial p}{\partial s_1}\right)_{\rho_1} = \frac{p}{c_{v1}} \tag{4.6}$$

An obvious compressibility influence is the change of fluid density. For the aerated water flow, Eq. (4.2) yields

$$\nabla\cdot\vec{u}_m = -\frac{1}{\rho_m}\frac{d_m\rho_m}{dt} \tag{4.7}$$

Thus the magnitude of the divergence of the mixture velocity may be used for determining the change of the mixture density and thus obtaining the compressibility criterion of the aerated water flow, *i.e.* the conditions under which the compressibility of the aerated water flow must be taken into account.

After some algebra operations of Eq. (3.12, 17), (4.3 – 6), Eq. (4.7) can be rewritten as

$$\nabla\cdot\vec{u}_m = \frac{1}{\rho_m a_m^2}\cdot\left\{\begin{aligned}&\frac{\rho_m}{2}\frac{\partial u_m^2}{\partial t} + \frac{C_{VM}(\gamma-1)\rho_m}{2}\frac{\partial u_r^2}{\partial t} - \frac{\partial p}{\partial t} - \rho_m\vec{u}_m\cdot\vec{g} + \\ &\frac{\rho_m}{2}\vec{u}_m\cdot\nabla u_m^2 + \frac{C_{VM}(\gamma-1)\rho_m}{2}\vec{u}_m\cdot\nabla u_r^2 + -C_{VM}\cdot \\ &(\gamma-1)\rho_m\vec{u}_r\cdot(\vec{u}_r\cdot\nabla\vec{u}_m) + \vec{u}_m\cdot[\nabla\cdot(\alpha\rho_1\vec{u}_r\vec{u}_r)] + \\ &C_D(\gamma-1)\rho_m(1-\alpha)^2u_r^3 + \frac{p}{c_{v1}}\vec{u}_r\cdot\nabla s_1\end{aligned}\right\} \tag{4.8}$$

where $u_m = |\vec{u}_m|$, $u_r = |\vec{u}_r|$.

Supposing there exist a characteristic velocity u_{m0} and a characteristic length l_0 in the considered problem and the magnitude of the change of the mixture velocity which occurs in the distance l_0 is u_{m0}, one can obtain the following criterion under which the compressibility mustn't be neglected: $\nabla \cdot \vec{u}_m \sim u_{m0}/l_0$. Under this condition the gradient of velocity must be dependent on the change of density and thus the change of density must have an important effect on the flow.

It is very convenient for us to use the dimensionless equations in the analysis of the order of the magnitude. The appropriate selection of the norm parameters can make the magnitude of each term of the dimensionless equations, not including the coefficient, be approximately 1, and thus the order of the magnitude of each term can be determined approximately by its coefficient. The whole list of the norm parameters for the present case is (u_{m0}, l_0, t_0, ρ_{m0}, a_{m0}, α_0, ρ_{10}, s_{10}, u_{r0}, g_0). Moreover, according to the momentum law, the norm parameter for pressure is $\rho_{m0}a_{m0}u_{m0}$, because the speed of the propagation of the pressure perturbation is identical with the sound velocity. We can then construct the corresponding dimensionless variables to replace those in Eq. (4.8) and obtain

$$\nabla^* \cdot \vec{u}_m^* = \frac{a_{m0}^2}{a_m^2} \cdot \left\{ \begin{array}{l} \dfrac{l_0}{a_{m0}t_0}\left[\dfrac{Ma_0^2}{2}\dfrac{\partial u_m^{*2}}{\partial t^*} + \dfrac{C_{VM}(\gamma-1)\delta_0^2 Ma_0}{2}\dfrac{\partial u_r^{*2}}{\partial t^*} - \dfrac{\rho_{m0}}{\rho_m}\dfrac{\partial p^*}{\partial t^*}\right] - \\ \dfrac{g_0 l_0}{a_{m0}^2}\vec{u}_m^* \cdot \vec{g}^* + \dfrac{Ma_0^2}{2}\vec{u}_m^* \cdot \nabla^* u_m^{*2} + \dfrac{C_{VM}(\gamma-1)\delta_0^2 Ma_0^2}{2}\vec{u}_m^* \cdot \\ \nabla^* u_r^{*2} - C_{VM}(\gamma-1)\delta_0^2 Ma_0^2 \vec{u}_r^* \cdot (\vec{u}_r^* \cdot \nabla^* \vec{u}_m^*) + \\ \dfrac{\alpha_0\rho_{10}}{\rho_m}\delta_0^2 Ma_0^2 \vec{u}_m^* \cdot [\nabla^* \cdot (\alpha^* \rho_1^* \vec{u}_r^* \vec{u}_r^*)] + \\ (\gamma-1)(1-\alpha_0)^2 C_D l_0 \delta_0^3 Ma_0^2 \left(\dfrac{1-\alpha}{1-\alpha_0}\right)^2 u_r^{*3} + \\ \dfrac{\rho_{m0}}{\rho_m}\dfrac{s_{10}}{c_{v1}}\delta_0 Ma_0 p^* \vec{u}_r^* \cdot \nabla^* s_1^* \end{array} \right\} \qquad (4.9)$$

Now the obvious compressibility criterion is $\nabla^* \cdot \vec{u}_m \sim 1$. In general, the aerated water flow can't be considered as incompressible if any term of the right side of Eq. (4.9) is of the order of 1 or more. The detailed analysis can be found in Zhao (1998). Here only the most common cases in which the compressibility influence must be taken into account are discussed.

1. Sound motion in the aerated water flow. For this case, l_0 is the wavelength, $1/t_0$ is the frequency, and thus $l_0/(a_{m0}t_0) \equiv 1$. Otherwise, $\rho_m/\rho_{m0} \sim 1$ and $a_m/a_{m0} \sim 1$. Then the coefficient of the term $\partial p^*/\partial t^*$ is of the order of 1, and thus all sound motions in the aerated water flow are compressible.
2. Aerated water flow with extra great depth. Under the normal condition, the sound velocity in the aerated water flow is about 30 m/s, g_0 = 9.8 m/s is the acceleration of the gravity. If $g_0 l_0/a_{m0}^2 \sim 1$ or $l_0 \sim a_{m0}^2/g_0 \approx$ 90m and the mixture velocity approximately parallels the gravity, the compressibility influence can not be neglected. It is the case of the aerated water flow in the impinging region in plumping pools.
3. Extra-high-velocity steady aerated water flow in which the mixture velocity has the same magnitude of the sound velocity. For this case, the term $\vec{u}_m^* \cdot \nabla^* u_m^{*2}$ is playing a dominant role. Its order is as the square of the adiabatic Mach number. Therefore, the adiabatic Mach number may act as the compressibility criterion for determining the importance of the compressibility of steady aerated water flow. All of the other terms contain the factor of the velocity-slip between the two phases, which also make some contribution to the compressibility of steady aerated water flow. This influence makes it difficult to determine the detailed compressibility standard of the general aerated water flow, although the value of the velocity-slip is small.

5 Detailed Compressibility Standard of Homogeneous Aerated Water Flow

For the steady homogeneous aerated water flow where $\partial/\partial t = 0$ and $\vec{u}_1 = \vec{u}_2 = \vec{u}_m$, the continuity equations of the air- and water-phase are

$$\nabla \cdot (\alpha\rho_1\vec{u}_m) = 0 \qquad (5.1)$$

$$\nabla \cdot [(1-\alpha)\rho_2\vec{u}_m] = 0 \qquad (5.2)$$

respectively. They yield

$$\alpha\rho_1/(1-\alpha) = \alpha_0\rho_{10}/(1-\alpha_0) = const. \qquad (5.3)$$

The energy equation of the air-phase Eq. (2.3) becomes

$$p/\rho_1^\gamma = p_0/\rho_{10}^\gamma = const. \qquad (5.4)$$

The relation of the mixture energy can be written as

$$h_m + \frac{u_m^2}{2} = h_{m0} = const. \qquad (5.5)$$

where the mixture enthalpy per unit mass is defined as

$$h_m = \frac{1}{\rho_m}[\alpha\rho_1 h_1 + (1-\alpha)\rho_2 h_2] \qquad (5.6)$$

For the temperature and the density of the water-phase remain constant throughout the flow, the enthalpy per unit mass of the water-phase h_2 also remains constant. For the sake of convenience, let $h_2 = 0$. Thus, using the air-phase state equation (2.4) and the relation $h_1 = e_1 + p/\rho_1 = c_{v1}T_1 + p/\rho_1$, it can be verified that

$$h_m = \frac{\gamma-1+\alpha}{\gamma-1}\frac{p}{\rho_m} \qquad (5.7)$$

and Eq. (5.5) can be rewritten as

$$\frac{\gamma-1+\alpha}{\gamma-1}\frac{p}{\rho_m}+\frac{u_m^2}{2}=\frac{\gamma-1+\alpha_0}{\gamma-1}\frac{p_0}{\rho_{m0}}=const. \tag{5.8}$$

Divided by the square of the Wood adiabatic sound velocity, we can obtain

$$Ma^2=\frac{2\alpha}{\gamma(\gamma-1)}\left\{(\gamma-1+\alpha_0)\frac{1-\alpha}{1-\alpha_0}\cdot\left[\frac{\alpha(1-\alpha_0)}{\alpha_0(1-\alpha)}\right]^{\gamma}-(\gamma-1+\alpha)\right\} \tag{5.9}$$

where the approximation $\rho_m \approx (1-\alpha)\rho_2$ has been used and then the relation (5.9) is tenable if the condition $(\alpha\rho_1/(1-\alpha)\rho_2) \ll 1$ is satisfied.

Using the relation (5.9), we can determine all parameters of the mixture flow as functions of the adiabatic Mach number *Ma*. Figure 2 shows the rate of the change of the air-water mixture density, *i.e.*

$$\frac{\Delta\rho_m}{\rho_{m0}}=\frac{|\rho_m-\rho_{m0}|}{\rho_{m0}},$$

as a function of the adiabatic Mach number *Ma*. It is shown that this rate is less than 4% when $Ma < 0.3$, and thus it is reasonable to consider the flow incompressible. On the other hand, when $Ma > 0.3$, the change of the mixture density is more remarkable, and the flow must be considered compressible. Therefore, the detailed compressibility standard of the steady homogeneous aerated water flow ought to read as follows:

$Ma < 0.3$, incompressible aerated water flow
$Ma > 0.3$, compressible aerated water flow

which is just the same as that in aerodynamics.

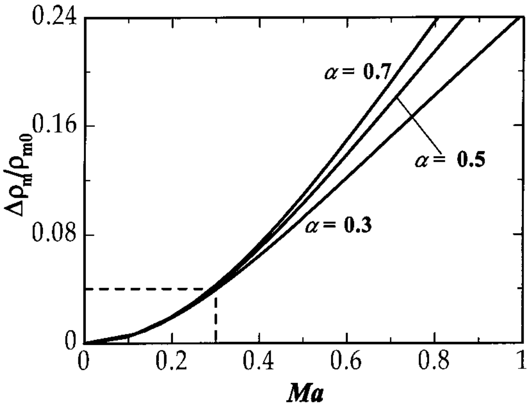

Fig. 2. The rate of the change of the air-water mixture density as a function of the adiabatic Mach number.

6 Concluding Remarks

Aeration is a common phenomenon of the high velocity water flow in the hydraulic engineering. The entrained air may dramatically enhance the compressibility of the aerated water flow. It is necessary and important to investigate the laws of the compressible aerated water flow in the hydraulic engineering, especially in the case of extra-great-height dam.

The present study has been focused on some fundamental topics of this problem. It has been shown that the Wood adiabatic formula is a good approximation for calculating the sound velocity in the extra-high-velocity aerated water flow. The compressibility influence of the extra-high-velocity aerated water flow in which the air-water mixture velocity is of the order of the sound velocity must be taken into account. A detailed compressibility standard has been obtained for the case of the steady homogeneous aerated water flow, which is just the same as that in aerodynamics.

7 Notations

a	sound velocity
c	specific heat
C_D	empirical parameter in the drag relation of Hench & Johnston (1972)
C_{VM}	added mass coefficient
d	air bubble diameter
e	internal energy per unit mass
$\vec{F}$	interfacial force
$\vec{g}$	body force (e.g. gravity)
h	enthalpy per unit mass
H	parameter defined as α'/ρ_1'
k	thermal conductivity
Ma	adiabatic Mach number
p	pressure
R	gas constant
T	temperature
$\vec{u}$	velocity

Greek Letters

α	concentration or air concentration
δ	relative velocity-slip
γ	ratio of specific heats
ρ	density
τ	characteristic time

Subscripts

ex	external
in	internal
k	*k*-phase
1	air-phase
2	water-phase
m	mixture
0	norm parameter

Superscripts

’ perturbation

* dimensionless variable

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