Solution for spillway flow by finite difference method
Solution des écoulements sur déversoir par la méthode de différences finies

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ABSTRACT

The stream function is used to analyze the irrotational flow over spillway crests, with and without gate. Items of interest include the free surface location and determination of the discharge and pressure fields. The approach, which is based on the finite difference method with a new representation of Neumann's problem on boundary points, is mathematically simple and requires only simple programming. Results thus obtained are in agreement with those obtained by way of experiments.

RÉSUMÉ

La fonction de courant est utilisée pour l'analyse de l'écoulement au dessus de la crête de déversoir, équipé ou non de vanne. Les buts en considération sont la détermination de la surface libre, le débit et le champs de pression. L'approche est basée sur la méthode de différences finies avec une nouvelle représentation de la condition de Neumann au contour, c'est mathématiquement simple et facile pour la programmer. Les résultats alors obtenus montrent une bonne concordance avec ceux qui ont été obtenus par la voie expérimentale.

Introduction

Escande [8] has shown, experimentally, that a two-dimensional flow over a spillway is irrotational. This sort of flow has attracted great interest because of its importance in several hydraulic fields. The problem of determining the flow over a spillway is, mainly, to obtain the contour of the free surface, the pressure distribution on the spillway and on the gate (if it exists), and, at last, the discharge when the upstream stagnant head is given. The difficulties of the problem are the nonlinear boundary conditions on the free surface and, according to the case, the flow is supercritical in the downstream range of the crest of the spillway, while it is subcritical in the upstream range.

Nevertheless, numerical solutions of problem have been found for many cases of practical interest. Chan et al. [5], Ikegawa and Washiu [12] and Diersh et al. [7] employed the variational principle by means of finite element method. Good convergence was encountered. However the accuracy is limited, especially in the pressure distribution, perhaps because of the use of linear element in the finite element solution. Li et al. [14] employed the finite analytic method to obtain the numerical solution. A boundary-fitted co-ordinate system is adopted to map the complex domain into a rectangular domain with uniform meshes. The discharge coefficient was deduced according to the variational principle for variable domains. Nevertheless, the accuracy of this study is, essentially, of the same order of magnitude as anterior studies.

To date, most of the existing finite difference methods can solve only problems with simple geometric boundary [1], [15]. The correction of the free surface is obtained either by manual correction or via a trial-and-error technique.

In the present study, a new approach by finite difference method is used. For a given mesh constant, the variable domains are discretized by straight lines parallel to the cartesian axes. These lines are intercepted by the boundary contour in points called boundary points. Only their co-ordinates are used to obtain the numerical solution. The great innovation is the Neumann boundary condition written in each point of the boundary. This fact yields excellent convergence and the accuracy is at least not smaller than that of the above anterior or similar methods. As well, the method is mathematically easy and simple to use. The resolution of the free surface is by successive approximation. The method could be used to analyze a wide variety of plane harmonic fields.

Finite-difference model

The flow is assumed to be steady, two-dimensional, incompressible and irrotational. Let the region Ω of the x, y-plane flow have a boundary C made up of one differentiable closed curve (or surface), that admits a parametric representation $x(s)$, $y(s)$, where s represents the arc length. Along a curve C, let the positive direction for s be in counter clockwise sense. Suppose that $x(s)$ and $y(s)$ may be expansible by Taylor series around any point P(s). Consequently, a right-handed system of triortogonal unit vectors $\vec{e}_s(s), \vec{e}_n(s), \vec{e}_b(s)$ can be assigned to any point P(s), where $\vec{e}_b(s)$ is normal to the x,y – plane at the positive direction of the Oz axis, $\vec{e}_s(s) \equiv (s_x, s_y)$ is tangential to the C curve at the positive s-direction, and $\vec{e}_n(s) \equiv (n_x, n_y)$ is such that $\vec{e}_n(s) = \vec{e}_b(s) \wedge \vec{e}_s(s)$, always inwards oriented, makes an angle $\pi/2$ advanced in relation to that of $\vec{e}_s(s)$. (s_x, s_y) and n_x, n_y are the directional cosines of $\vec{e}_s(s)$ and $\vec{e}_n(s)$, respectively. For a real potential function $f(x, y)$, defined in Ω, denote by $\nabla^2 f(x,y)$ the Laplacian operator $\partial^2 f / \partial x^2 + \partial^2 f / \partial y^2$. A fairly common class of spillway flow is included in the following problem, that is, in the solution of Laplace's equation

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$$\nabla^2 f(P) = 0, \qquad P \text{ in } \Omega \tag{1}$$

subject to the mixed boundary conditions

$$f(P) = g_1(P), \qquad \text{P on } C_1 \text{ contained in C}, \tag{2}$$

$$\frac{\partial f(P)}{\partial n} = g_2(P), \qquad \text{P on } C_2 \text{ contained in C}, \tag{3}$$

where g_1 and g_2 are known functions in P on C and $\partial f(P)/\partial n$ denotes the derivative of $f(P)$ in the inwards normal direction (Neumann's condition).

In numerical computations Ω, C, f and $\nabla^2 f$ are frequently replaced by corresponding quantities associated with a finite difference network, introduced as follows (Fig. 1): For a given mesh constant $h_m > 0$, let a net consist of straight lines given, by $x_i = ih_m$, $y_j = jh_m$ $i,j = 0, 1, 2, ...$, respectively, which determine on C the points $P_k = (x_k, y_k)$, $k = 1, 2, ...$, called boundary points of the net. The points $P_{i,j} = (ih_m, jh_m)$ in Ω are nodes. The $P_{i,j}$ nodes and the P_k boundary points form a net assumed to be connectible by straight segments of the net within Ω. A point $P_{i,j}$ is a regular interior point if it is connected with four neighboring nodes $(ih_m \pm h_m, jh_m \pm h_m)$. All other points $P_{i,j}$ connected with regular interior points and boundary points P_k are irregular interior points. Under finite difference form, the Eqs. 1,2,3 can be written as follows:

For simplicity, let each irregular interior node $P_{i,j}$ be denoted by 0 and the neighboring nodes on C and in Ω by 1,2,3,4. Let the segment length be denoted *a, b, c, d*. It results from Eq.1, [9],

$$r_0 = \frac{f_1}{a(a+c)} + \frac{f_2}{b(b+d)} + \frac{f_3}{c(c+a)} + \frac{f_4}{d(d+b)} - \frac{ac+bd}{abcd} f_0 \tag{4}$$

where r_0 is the residual at 0 and each variable of type $f_w = f(P_k)$, $1 \le w \le 4$, is either given by Eq.2 or determined by Eq.3. If 0 is a regular interior node, Eq.4 reduces to the usual formula

$$r_0 = f_1 + f_2 + f_3 + f_4 - 4 f_0 \tag{5}$$

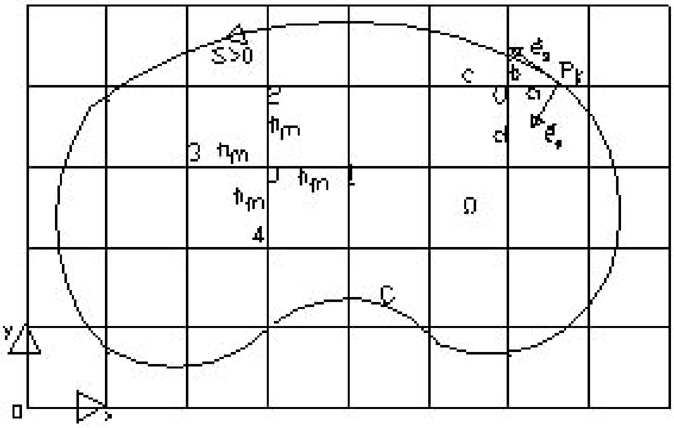

Fig. 1 Closed curve C as boundary of region Ω in parametric representation with s-direction counter clockwise, unit tangential and normal vectors, mesh constant h_m, regular and irregular interior points, and Cartesian axes with origin in the lower left corner of the net.

Following [9], when certain conditions are satisfied, $r_o \cong 0$, thus Eq.4 and Eq.5 form a system of equations equivalent to Eq.1. The Neumann condition (Eq.3) will be introduced by the formula (see Appendix), at points P_k which lie on horizontal mesh lines:

$$f_k = f_o + n_x \left(\frac{\partial f}{\partial n}\right)_k \delta x - s_x n_x \left[\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial n}\right)\right]_k \delta x^2 + s_x \left(\frac{\partial f}{\partial s}\right)_k \delta x - \frac{1}{2}\left(s_x^2 - n_x^2\right)\left(\frac{\partial^2 f}{\partial s^2}\right)_k \delta x^2 \tag{6a}$$

and on vertical mesh lines:

$$f_k = f_o + n_y \left(\frac{\partial f}{\partial n}\right)_k \delta y - s_y n_y \left[\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial n}\right)\right]_k \delta y^2 + s_y \left(\frac{\partial f}{\partial s}\right)_k \delta y - \frac{1}{2}\left(s_y^2 - n_y^2\right)\left(\frac{\partial^2 f}{\partial s^2}\right)_k \delta y^2 \tag{6b}$$

where $\delta x = x_k - x_o$ and $\delta y = y_k - y_o$

$\frac{\partial f}{\partial n}$ and $\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial n}\right)$ should be given, while $\frac{\partial f}{\partial s}$ and $\frac{\partial^2 f}{\partial s^2}$ must be determined by the finite difference method when unknown.

The system of linear equations is sparse, that is, most of the coefficients in each equation are zero. Since the diagonal terms are dominant, that is, they are at least as large as the sum of the off-diagonal terms [+4 or $(ac+bd)/abcd$] and actually larger in at least one case, the system can be solved by the Gauss-Seidel iteration method, which converges to a solution of Eq.1. This method is marked by its simplicity and the ease with which it may be programmed for a computer.

To test the accuracy of the method for a domain where only Neumann's condition is given, an example worked by Southwell and Vaisy [16] was chosen. This has a known theoretical solution, simple to compute, and entails rapid variation of $\partial f/\partial n$ in some part of the boundary. The result revealed an excellent accuracy, [2].

Formulation of the problem

Fig.2 shows a typical region of two-dimensional flow over a spillway. Let Ω indicate the flow region. Thus the boundary contour C will be formed by the surfaces S_1 of the reservoir (or channel) bed and spillway, the free surface S_3, which is unknown beforehand, and the far upstream and downstream sections, denoted by S_4 and S_2, which are assumed to be normal to the reservoir bed and the spillway surface, respectively. The origin O(0,0) of the co-ordinate system is always fixed in the left inferior corner and the Ox, Oy axes are taken in the horizontal and vertical directions, respectively. Assuming the total head is given, the stagnant level can be located H above the datum $y=0$. In this case, the flow rate for the unit width of the spillway q is unknown.

Assuming an irrotational flow is in Ω, then the stream function f

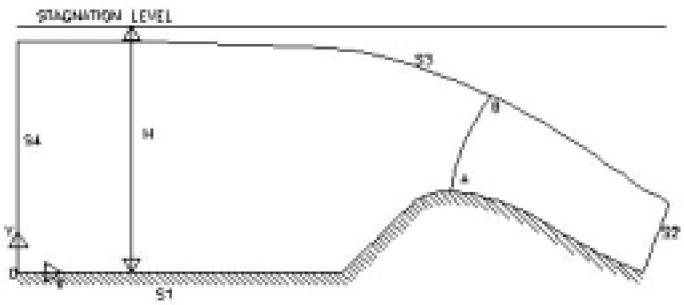

Fig. 2 Spillway flow with known stagnation level, free surface S_3 preliminarily unknown and AB critical section from known topmost spillway sole A to B.

= $\psi(x, y)$ is plan-harmonic governed by the Eq.1. Moreover, if v_s and v_n denote the respective components of the velocity <?> in the directions of $\vec{e}_s$ and $\vec{e}_n$ at any point of C, $\psi(x,y)$ can therefore be defined by:

$$\frac{\partial\psi}{\partial s} = v_n, \qquad \frac{\partial\psi}{\partial n} = -v_s, \tag{7}$$

which will automatically satisfy the equation of continuity for an incompressible two-dimensional flow. Moreover with every solution ι of Eq.1 there is associated a conjugate plan-harmonic function φ, called velocity potential function, such that

$$\frac{\partial\varphi}{\partial s} = \frac{\partial\psi}{\partial n}, \qquad \frac{\partial\varphi}{\partial n} = -\frac{\partial\psi}{\partial s}. \tag{8}$$

Besides, the velocity v, the pressure p and the elevation y are related by the Bernoulli equation
vz

$$\frac{v^2}{2g} + \frac{p}{\rho g} + y = H, \tag{9}$$

valid for every point in Ω. Let p henceforth be a gage pressure. At the free surface, therefore, this pressure vanishes and the velocity, at any point of that boundary, follows from Eq.9

$$v = \sqrt{2g(H - y)} \tag{10}$$

where $y=y(x)$ is the elevation of the point.

The critical variables. The effects of gravity on the liquid motion when the free surface is disturbed depend on the dimensionless Froude number $F_r = v_o / \sqrt{gL}$, where v_o is the flow velocity, L a caracteristic length, and $c = \sqrt{gL}$ is the wave speed relative to the liquid at which infinitesimal disturbances are propagating along the surface. Let the flow speed V, the depth of flow Y, and the wave speed $c = \sqrt{gY}$ at a cross section of the channel be used in defining the Froude number $F_r = V / \sqrt{gY}$ at section. A flow with $V>c$ ($F_r>1$), V=c (F_r=1), and V<c (F_r<1) is said to be supercritical, critical or subcritical, respectively. The flow from F_r<1 to F_r>1 is accelerated, necessarily continuous, and its passage is called critical cross section, where F_r=1. The water depth Y_{cr} corresponding to F_r=1 is called critical depth. By different ways, Guo et al. [10] and Brun et al. [4] proved that the critical cross section is an equipotential line (φ_{AB}=const.) passing the highest point A on the spillway and intercepting the free surface at point B, called critical point. To the critical depth corresponds a minimum specific energy whose value is

$$H_s = H - y_A = y_B - y_A + v_B^2/2g \tag{11}$$

The sole of the spillway is, in general, upward (positive slope) in the upstream range of the crest and downward (negative slope) in the downstream range. These slopes of different signs of the sole can yield the following well known effects [4]: if the flow is only subcritical along the spillway, the velocity increases and the depth decreases in the upstream side and the velocity decreases and the depth increases in the downstream side; if the flow is only supercritical, all the variation of direction changes: the velocity decreases and the depth increases in the upstream side and the velocity increases and the depth decreases in the downstream side. So if a flow over a hydraulic structure (spillway preceded by reservoir or channel) with a slope of contrary signs has an always increasing velocity and the problem has an unknown boundary part to be obtained by successive approximation (case of the Fig.2), the boundary condition needs to be absolutely compatible with the subcritical flow in the upstream region and with the supercritical flow in the downstream region. Dirichlet's condition was found adequate for the subcritical flow and Neumann's condition, for the supercritical flow.

The gradient from numerical analysis. Some values of type $\partial\psi/\partial n$ at the points of the boundary stream line of the net where ψ is given, must be found from the numerical solution of Eq 1. The method of calculation makes use of the fact that, along a free stream line,

$$\frac{\partial\psi}{\partial s} = 0,$$

and of the gradient expressions

$$\nabla\psi = \frac{\partial\psi}{\partial x}\vec{e}_x + \frac{\partial\psi}{\partial n}\vec{e}_y = \frac{\partial\psi}{\partial s}\vec{e}_s + \frac{\partial\psi}{\partial n}\vec{e}_n.$$

Whence,

$$\frac{\partial\psi}{\partial n} = \frac{1}{n_x}\cdot\frac{\partial\psi}{\partial x}, \qquad \frac{\partial\psi}{\partial n} = \frac{1}{n_y}\cdot\frac{\partial\psi}{\partial y}, \tag{12}$$

at points which lie on horizontal and vertical mesh lines, respectively.

The pressure head distribution. The first application of Eqs.12 is the calculation of pressure at points of spillway and gate, when the definitive numerical solution of Eq.1 is obtained. Since the velocity at those points, according to the Eq.7, is given by $v=\pm\partial\psi/\partial n$ the pressure head follows from Eq.9

$$h_p = \frac{p}{\rho g} = H - \frac{1}{2g}\left(\frac{\partial\psi}{\partial n}\right)^2 - y \qquad (13)$$

The boundary conditions. To establish the boundary conditions along the contour C of the flow region, let $y^{(i)}$, $x^{(i)}$, $\psi^{(i)}$ and $q^{(i)}$, respectively, denote the co-ordinates, the stream function, the velocity and the estimated discharge (see below) for the approximation i. The boundary conditions are given as follows:

1. Since S_1 should be a fixed stream line, <?> must be a constant whose value can be considered as the origin of the flow rate. Therefore, the Dirichlet condition is used for the numerical analysis

$$\psi^{(i)}=0 \qquad (14)$$

2. The free surface S_3 is initially unknown and two boundary conditions need to be satisfied there, as well as the discharge $q^{(i)}$ obtained by successive approximations. The critical point B is located, roughly, on the point immediately at right of point A. Along S_3, from point B to the upstream region, the Froude number is $F_r \leq 1$, therefore the Dirichlet condition is used for the numerical analysis

$$\psi^{(i)} = q^{(i)}. \qquad (15)$$

From point B to the downstream region, the flow is supercritical, therefore the Neumann condition can be used. Since $v_s = -v^{(i)}$ and $v_n = 0$, it follows, from Eq.7 and Eq.10, that $\partial\psi/\partial s = 0$ and $= \partial\psi/\partial s$ $v^{(i)} = \sqrt{2g(H-y^{(i)})}$. Eq.6 becomes, at points $P_{k,}$ which lie on horizontal mesh lines:

$$f_k = f_o + n_x(v^{(i)})_k\delta x - s_x n_x\left(\frac{\partial v^{(i)}}{\partial s}\right)_k\delta x^2 \qquad (16a)$$

and on vertical mesh lines,

$$f_k = f_o + n_y(v^{(i)})_k\delta y - s_y n_y\left(\frac{\partial v^{(i)}}{\partial s}\right)_k\delta y^2; \qquad (16b)$$

where $\delta x = (x^{(i)})_k - x_0$ and $\delta y = (y^{(i)})_k - y_0$.

The other boundary condition is used to obtain the adjustement of the free surface.

3. Let S_2 and S_4 be equipotential lines (normal to the all stream lines among S_1 and S_3). Since $v_s=0$ and $v_n=-v$ on S_2 and $v_n=v$ on S_4, it follows, from Eq.7 and Eq.10, that $\partial\psi/\partial s = v_n$ and $\partial\psi/\partial n = 0$. Hence,

$$\frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial n}\right) = 0, \quad \frac{\partial^2\psi}{\partial s^2} = \frac{\partial v_n}{\partial s}.$$

Eq.6 becomes, at points $P_{k,}$ which lie on horizontal mesh lines:

$$f_k = f_o + s_x(v_n)_k\delta x - \frac{1}{2}(s_x^2 - n_x^2)\left(\frac{\partial v_n}{\partial s}\right)_k\delta x^2, \qquad (17a)$$

and on vertical mesh lines:

$$f_k = f_o + s_y(v_n)_k\delta y - \frac{1}{2}(s_y^2 - n_y^2)\left(\frac{\partial v_n}{\partial s}\right)_k\delta y^2; \qquad (17b)$$

where $\delta x = x_k - x_o$ and $\delta y = y_k - y_o$ Since the velocity is uniform on S_4, then $v_n=v=\text{const}$ and $\partial v_n/\partial s = 0$.

The approximation model. The adjustment of the free surface in each of its points P_k to the next iteration from that used in the present calculation can be expressed by:

– Following Eq.7, the condition that must occur on the upstream region of the free surface has to be $\partial\psi/\partial n = v^{(i)}$, where $\partial\psi/\partial n$ is given by Eq.12 and $v^{(i)}$, by Eq.10. In case of differences between these values, the elevation to the next approximation is

$$y^{(i+1)} = H - \frac{1}{2g}\left(\frac{\partial\psi^{(i)}}{\partial n}\right)^2, \qquad (18)$$

obtained according to Eq.10 and Eq.12.

– The condition that must occur on the downstream region of the free surface is $df \cong q^{(i)} - \psi^{(i)} = 0$. In case of differences, the correction $dn^{(i+1)}$ following the normal direction to the free surface should be

$$q^{(i)} - \psi^{(i)} \cong df = \left(\frac{\partial\psi^{(i)}}{\partial n}\right)dn^{(i+1)} = v^{(i)}.dn^{(i+1)}$$

where, from Eq10, $v^{(i)} = \sqrt{2g(H-y^{(i)})}$. Whence

$$dn^{(i+1)} = \frac{q^{(i)} - \psi^{(i)}}{v^{(i)}} \qquad (19a)$$

and the co-ordinates of the next approximation are given by

$$x^{(i+1)} = x^{(i)} + n_x dn^{(i+1)}, \quad y^{(i+1)} = y^{(i)} + n_y dn^{(i+1)} \qquad (19b)$$

The numerical solution. To begin with, a preliminary value of the discharge q is assumed and the cross sections S_2 and S_4 are determined according to continuity. The procedure is as follows:

1. Assume the free surface for the approximation i=1.
2. Solve the problem so many times as necessary, changing the flow rate each time until its value becomes equal to the value of ψ obtained at the boundary point P_{k-1} immediately anterior to the critical point $P_{k.}$ In that moment, the flow rate will be $q^{(i)} = (\psi^i)_k$.
3. Adjust the free surface according to Eq.18 and Eq.19a andrepeat step 2 successively, until obtaining on all points P_k of the free surface and the flow rate as

$$\left|y^{(i+1)} - y^{(i)}\right| < \varepsilon_y, \qquad \left|dn^{(i+1)}\right| < \varepsilon_n.$$

where ε_y and ε_n are absolute errors.

4. Use the Eq.13 to obtain the pressure head distribution on the spillway.

So $q^{(i+1)} \cong q^{(i)}$ is the discharge, which will be, evidently, unless there is a small error, of the same order of the flow rate value evaluated at cross sections S_4 by the formula

$$q = h\sqrt{2g(H_o - h)} \tag{20}$$

where h is the depth and H_o is the specific energy at S_4.

Applications

Three geometries of spillways are considered to test the applicability and versatility of the aforementioned method.

The circular spillway. Fig.3 shows a spillway to consist of a quarter of circle preceded by a horizontal bed channel. Details of the spillway geometry and experimental measurements were taken from Böss, [3]. The basic data of the spillway and flow are: radius, 20cm, flow rate for one meter width, q_o = 80 *l*/s.m (800 cm^3/s.cm), the critical depth, h_{cr}=8.68cm, and the stagnant level is located H_o =13.02 cm above the channel bed

To be data compatible, the gravitational acceleration was taken g=978.64 cm/s^2 and h_m=1cm to the mesh constant. It is easy now to verify that h_{cr} is the far upstream depth in the channel, q_o is the maximum discharge and H_o is the minimum specific energy. The flow through cross section S_4 was considered uniform, then $\nu_{(s)}$=ν_A= const., and through S_2, the stream lines were supposed to be circular concentric lines, where the velocity is given by $\nu_{(s)}r_{(s)}$=const., [13]. Ignoring the aforementioned data, the flow over the spillway was studied beginning with h>h_{cr} (h=10cm was assumed) at S$_4$, whence q is given by Eq.20, S_2 is determined by continuity and a free surface is assumed between S_2 and S_4. Since there is no positive slope and the velocity is always increasing along the free surface from S_4 to S_2, a unique boundary condition is then sufficient for every point of the free surface. The Neumann condition, given by the Eq.16, was chosen and Eq.19 was used for the correction of the free surface. The convergence was relatively fast. During the convergence process, the free surface was always converging. When h arrived at h_{cr}, the flow rate q was substituted by the value ψ_X at point X to maintain this point unchangeable henceforth and next to proceed until the convergence of the free surface. This surface was considered solution when the maximum dn=(ψ_X - ψ)/ν obtained was 0.46mm. For this accuracy six iterations were required. Fig.3 shows excellent agreement of the free surface with the measured data. The maximum flow rate encountered along the solution free surface was ψ_o=811.998 cm^3/s.cm, therefore the maximum relative error for the discharge is $\varepsilon_q = (\psi_0 - q_0)/q_0$ = 0,015 or 1,5%. Fig.3 shows still a small discrepancy between the predicted pressure line and that measured. Considering the wide extension of the curve of the spillway and of the free surface and h$_m$ that is not sufficiently small, the results obtained seem to be encouraging and, particularly, show that the Eq.16 and Eq.19 are well adequate to solve the Neumann condition by successive approximation method.

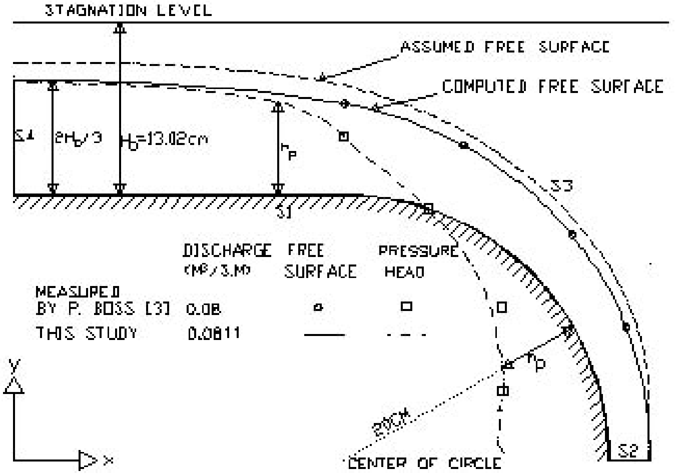

Fig. 3 Quarter circle spillway following horizontal bed. Measurements on model by P. Böss [3] compared with predicted values in this study.

The Itaipu spillway. To show the performance of the method in hydraulic structures of practical interest, this study included the flow over a spillway, with and without a sluice gate. The geometry of the spillway chosen is the *Itaipu* dam spillway (located on the river Parana and owned by Brazil and Paraguay). Details of the spillway geometry are given by De Moraes et al. [6].

Fig.4 displays some aspects of the spillway geometry, of the data assumed and the performance of the flow. The boundary conditions are the same aforesaid for Fig.2. The downstream extremity of the profile S_1 is an arc of circle and the cross section S_2 is located following the radius of the circle, which can be justified as an equipotential line and treated as the anterior case. The discharge was found equal to 199.0 m^3/s.m.

Fig.5 displays the same spillway and the flow under a radial gate. The boundary conditions are the same as above discounting the following differences: Eq.15 and adjustment by Eq.18 along the upstream portion S_3 of the free surface; Eq.15 along the gate (S_5); Eq.16 and adjustment by Eq.19 along the downstream portion S_3 of the free surface; the summit Y of the gate is at the same time a stagnation point of the free surface, where the elevation must equal H and $\nu_y = (\partial\psi/\partial n)y$=0; since the bottom G of the gate is a fixed point, belonging at the same time to the gate, where the velocity $\nu_g = (\partial\psi/\partial n)g$ changes for each corresponding q, and to S_3, where the velocity must be constant and equal to $v_G = \sqrt{2g(H - y_G)}$; in this case, the procedure for each alternative of the calculation

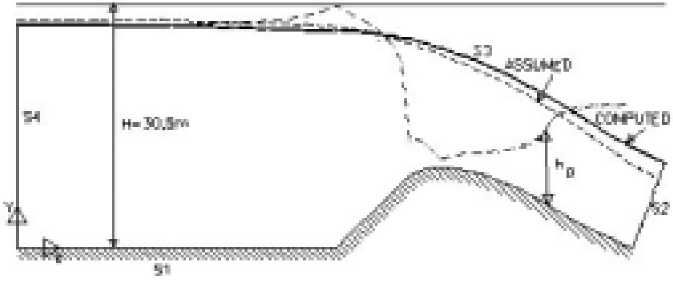

Fig. 4 Solution for the spillway flow of *Itaipu Binacional* plant.

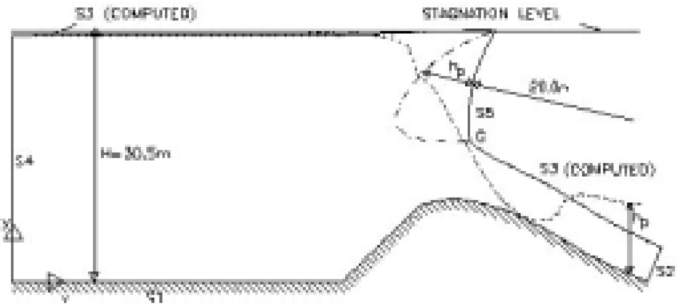

Fig. 5 Solution for the same spillway flow of *Itaipu Binacional* plant with radial gate.

is to vary a guessed value of q step by step, until finding a most appropriate value of q for assuring the equality of both velocities. After that, the S_3 are adjusted for the next alternative. Very good convergence of the solutions was obtained. The discharge was found equal 97.50 m^3/s.m.

Conclusion

In this study, the flow over spillways, with and without a sluice gate, has been solved by the finite difference methods, thanks, in particular, to a good and simple representation of the Neumann condition on the boundary contour. The governing system of equations is linear, sparse, and can be efficiently solved by the Gauss-Seidel iteration method.

Comparisons of the results obtained for circular spillway with results obtained experimentally indicate that the method's accuracy is very satisfactory.

The method could be used to analyze a wide variety of problems, as long as the plane harmonic theory is valid, and requires only easy programming.

Acknowledgements

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Appendix

The Eq.6 can be proved as follows (see [11] for a similar treatment):

The value of function f_k at those points of the contour C where it is cut by horizontal mesh lines can be evaluated approximately by (Fig.1),

$$f_k = f_o + (\frac{\partial f}{\partial x})_k dx - \frac{1}{2}(\frac{\partial^2 f}{\partial x^2})_k dx^2 \tag{A1}$$

with neglect of all the terms containing third or higher powers of $dx \cong x_k - x_o$. The first derivative may be written as

$$\begin{aligned}(\frac{\partial f}{\partial x})_k &= \vec{e}_x \times (grad f)_k = \vec{e}_x \times (\frac{\partial f}{\partial s}\vec{e}_s + \frac{\partial f}{\partial n}\vec{e}_n)_k = \\ &(\frac{\partial f}{\partial s}s_x + \frac{\partial f}{\partial n}n_x)_k\end{aligned} \tag{A2}$$

From this equation, the second derivative follows

$$\begin{aligned}(\frac{\partial^2 f}{\partial x^2})_k &= \vec{e}_x \times (grad\frac{\partial f}{\partial x})_k = \vec{e}_x \times [grad(\frac{\partial f}{\partial s}s_x + \frac{\partial f}{\partial n}n_x)]_k = \\ &= s_x[\frac{\partial}{\partial s}(\frac{\partial f}{\partial s}s_x + \frac{\partial f}{\partial n}n_x)]_k + n_x[\frac{\partial}{\partial n}(\frac{\partial f}{\partial s}s_x + \frac{\partial f}{\partial n}n_x)]_k = \\ &= (\frac{\partial^2 f}{\partial s^2})_k s_x^2 + 2[\frac{\partial}{\partial s}(\frac{\partial f}{\partial n})]_k s_x n_x + (\frac{\partial^2 f}{\partial n^2})_k n_x^2,\end{aligned} \tag{A3}$$

whence

$$(\frac{\partial^2 f}{\partial x^2})_k = (\frac{\partial^2 f}{\partial s^2})_k (s_x^2 - n_x^2) + 2[\frac{\partial}{\partial s}(\frac{\partial f}{\partial n})]_k s_x n_x,$$

in supposing the Schwarz condition $\frac{\partial}{\partial n}\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\frac{\partial f}{\partial n}$ is verified and $\nabla^2 f$ is invariant by rotation of the co-ordinate frame. Eq.6a is obtained substituting from Eq.A2 and Eq.A3 for the first and second derivatives into Eq.A1. In the same way, Eq.6b may be obtained.

References

[1]. ALLEN, D. N. de G., (1954), Relaxation Methods, McGraw-Hill, New-York, N.Y..

[2]. ASSY, T. M., and MARCICANO, J. P. P., (1993), Uma nova representação da condição de Neumann, pelo método das diferenças finitas, em campos harmônicos planos, 12th Brazilian congress of mechanical engineering, Brasilia-Brazil.

[3]. BÖSS, P., (1929), Berechnung der Abflussmengen und der Wasserspiegellage bei Abstürzen und Schwellen unter besonderer Berücksichtigung der dabei auftretenden Zusatzspannungen, Wasserkraft und Wasserwirtschaft, Vol. 24, 28-33.

[4]. BRUN, E. A., MARTINOT-LAGARDE, A., and Mathieu, J., (1960), Mécanique des Fluides, Tome I, Dunod, Paris.

[5]. CHAN, S. T. K., LAROCK, B. E., and HERRMANN, L. R., (1973), Free-surface ideal fluid flows by finite elements, Journal of the Hydraulics Division, ASCE, No. HY6, 959-974

[6]. DE MORAES, J., VILLALBA, J. R, DE MELLO, W. F., POLY, L. C., BERNY, O., ACOSTA, G., and SARKARIA, G. S., (1979), Selection of basic design of Itaipu spillway, 13 ICOLD, Congress New Delhi, Q50(R14), 249-272.

[7]. DIRSH, H. J., SCHIRMER, A., and BUSH, D. F., (1977), Analysis of Flows with Initially Unknown Discharge, Journal of the Hydraulics Division, ASCE, Vol. 105, 213-251.

[8]. ESCANDE, L., (1937), Barrages, in C. Camichel (ed.), "Hydraulique", III, Hermann & Cie, Paris.

[9]. FORSYTHE, G. E., and WASOW, W. R., (1967), Finite-difference methods for partial differential equations, John Wiley, NewYork.

[10]. GUO, Y. K., WEN, S. Y., WU, C. G., and FANG, D., (1998), Numerical modelling of spillway flow with free drop and initially unknown discharge, Journal of Hydraulic research, IAHR, Vol. 36, 785-801.

[11]. HUARD DE LA MARRE, P., (1956), Résolution de problèmes d'infiltrations a surface líbre au moyen d'analogies électriques, Thèse, Faculté des Sciences de l'Université de Paris.

[12]. IKEGAWA, M., and WASHIZU, K., (1973), Finite element method applied to analysis of flow over a spillway crest, International Journal for Numerical Methods in Engineering, Vol. 6, 179-189

[13]. LAUFFER, H., (1936), Ecoullement dans les canaux à radier incurvé, Wasserkraft und Wasserwirtschaft, (Traduction Chatou).

[14]. LI, W., XIE, Q., and CHEN, C. J., (1989), Finite Analytic Solution of Flow over Spillways, Journal of Engineering Mechanics, Vol. 115, No. 12, 2035-2648.

[15]. SOUTHWELL, R. V., and VAISEY, G., (1946), Relaxation methods applied to engineering problems, XII, Fluid motions characterized by "free" stream-lines, Phil. Trans. Roy. Soc. A, 240, 117-161.

[16]. SOUTHWELL, R. V., and VAISEY, G., (1943), Relaxation methods applied to engineering problems, VIII, Plane-potential problems involving specified normal gradients, Proc. Roy. Soc. A. 182, 129-151

Notations

ψ	stream function
φ	velocity potential function
Ω^*	flow region
C	closed curve (or surface)
x,y	rectangular cartesian co-ordinates
s	arc length
$\vec{e}_s, \vec{e}_n, \vec{e}_b$	a right-handed ortogonal system of unit vectors at each point of C
s_x, s_y	directional cosines of $\vec{e}_s$
n_x, n_y	directional cosines of $\vec{e}_n$
i,j,k	subscripts used to designate points $P_{i,j}$ in Ω and P_k on C
h_m	mesh constant
$\partial f / \partial n$	the inwards normal derivative
$(\ )k$	a value due to the point P_k
$(\ )^{(i)}$	a value for the approximation i
H	total head
H_o, H_s	specific energy
ν	velocity
ν_s, ν_n	tangential and normal components of the velocity, respectively
p	gage pressure
g	gravitational acceleration
ρ	mass density
q, q_o	discharge for the unit width of the spillway
$\varepsilon_y, \varepsilon_n$	absolute errors
×	sign of scalar product of two vectors
Λ	sign of vector product of two vectors