Structure of the turbulent hydraulic jump in a trapezoidal channel

Structure du ressaut hydraulique turbulent dans un canal trapézoïdal

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ABSTRACT
The axial flow structure of turbulent hydraulic jump has been analysed and the general equation valid for a channel of arbitrary cross section has been proposed. Based on the Reynolds equations of mean turbulent motion in two dimensional steady incompressible flow subjected to hydrostatic pressure distribution, the integral equations of depth averaged flow over a channel of arbitrary cross sectional area are obtained. An integral method has been developed where inertia, pressure gradient and depth averaged normal Reynolds stress play the dominant role. The closure model for variation of depth averaged normal Reynolds stress has been expressed as product of the constant eddy viscosity and the gradient of the depth averaged axial velocity with respect to axial distance. In the trapezoidal channel the closed form solution for the upper surface profile and axial length of the hydraulic jump have been obtained. The comparison of the theory with experimental data is remarkably good. The theory shows that for F_1 larger than a fixed value, the surface profile approaches a limiting universal solution provided the variables are appropriately non-dimensionalized. Further, the present predictions on the roller length are also supported by experimental data in rectangular and triangular channels.

RÉSUMÉ
La structure de l'écoulement axial dans un ressaut hydraulique turbulent a été analysée et l'on propose une équation générale aplicable à un canal de section quelconque. A partir des équations de Reynolds pour le mouvement turbulent moyen dans un écoulement bidimensionnel incompressible, soumis à une distribution de pression hydrostatique, on obtient les équations intégrales de l'écoulement moyenné sur la hauteur, dans un canal de section quelconque. Une méthode intégrale a été développée, dans laquelle l'inertie, le gradient de pression et la contrainte de Reynolds normale moyennée en hauteur, jouent le rôle principal. Pour fermer le modèle, la contrainte de Reynolds normale moyenne est exprimée comme le produit d'une viscosité turbulente constante par le gradient axial de la vitesse axiale moyennée en hauteur. Dans le canal trapézoïdal, on a obtenu le profil supérieur de la surface libre et la longueur axiale du ressaut hydraulique. La comparaison des résultats théoriques avec les données expérimentales est remarquablement bonne. La théorie montre que pour F_1 plus grand qu'une valeur donnée, le profil de la surface tend vers une solution limite universelle lorsque les variables sont convenablement adimensionnalisées. De plus, les résultats concernant la longueur de rouleau sont également confirmés par les données expérimentales en canaux rectangulaires et triangulaires.

1. Introduction

In a hydraulic jump the supercritical flow changes to subcritical flow with a rapid rise of flow depth associated with surface air entrainment. The air-water interaction begins at the toe of the jump that produces surface rollers on upper fluid surface, as displayed in Fig 1. The longitudinal and transverse length scales of the mixing layer due to air-water interaction depend on the upstream Froude number F_1, in addition to the Reynolds number (Long et. al 1991). The turbulent hydraulic jump is one of the oldest and hardest problem in hydraulic engineering. It has been studied extensively, but only a few investigators considered the internal structure of the flow in the jump and there is scarcity of analytical studies as described in the reviews of Rajaratnam (1967) and Hager (1992). The rectangular channel data have been analysed by Bakhmeteff and Matzke (1936), Schroder (1963, see Hager 1992), Gupta (1967), Rajaratnam and Subramanya (1968) and Sarma and Newnham (1973) that displayed the similarity of the surface profile for $F_1 \geq 4$. The non-dimensional variables proposed in each work are however different.

The bottom surface of the channel does not play a predominant role, except that the skin friction at the bottom wall affects the physical location of the hydraulic jump. The flow in the hydraulic jump, above the bottom surface, has been regarded as a turbulent shear layer having an air-water interaction on the upper surface, thus forming the rollers in the mixing layer, the extent of which depend on the magnitude of the upstream Froude number. The analysis of the hydraulic jump in a rectangular channel employed the standard turbulent boundary layer equations subject to hydrostatic pressure distribution. The depth averaged equations were obtained, which failed to account for the loss of specific energy. Thus, an additional equation, described below, is needed.

1) The depth averaged energy equation of the mean flow where sudden loss of the specific energy is represented by the production term was considered by Tsubaki (1950) based on the eddy viscosity model, and Madsen & Svendsen (1983) based on the κ-ε model.

2) Valiani (1997) considered an additional equation from depth average angular momentum balance between turbulent stress,

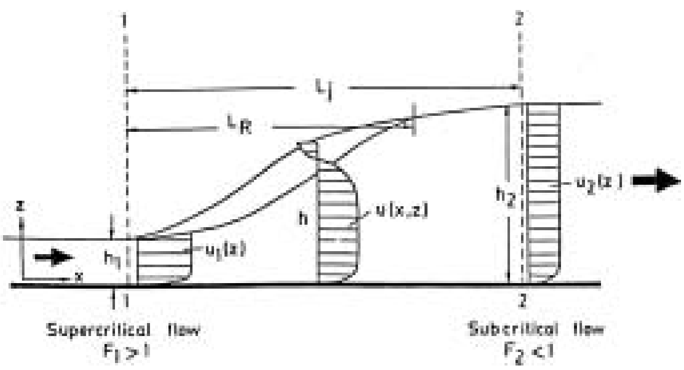

Fig. 1 Schematic diagram of typical hydraulic jump.

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vertical momentum and moments due to pressure force and linear horizontal momentum.

The attempt to modify the depth averaged streamwise momentum equation is considered by Rahman and Chaudhry (1995), which includes additional terms due to the energy grade line slope (as a function of Manning coefficient), the slope of bottom surface and Boussinesq terms (containing the third order derivatives of velocity) but these effects were not found important. Consequently, the Reynolds equations of two dimensional mean turbulent motion, with κ-ε model hypothesis have been numerically studied by Long et. al (1990) for forced flow and by Liu and Drewes (1994) in free and forced flows of hydraulic jump.

In the trapezoidal channel the experimental investigations on the hydraulic jump have been reported by Posey and Hsing (1938), Sandover and Holmes (1962), Press (1961), Mohed and Sharp (1971), Ohtsu (1976), Ali and Ridgway (1977) and Wanoschek and Hager (1989). Posey and Hsing (1938) expressed the average length as

$$\frac{L_j}{h_2} = 5\left[1+4\left(\frac{W_2-W_1}{W_1}\right)^{\frac{1}{2}}\right] \quad (1)$$

where $W=b+2nh$ is the water surface width, b is the bottom width, n is the slope of side walls and h is the axial depth of flow. The scatter of data is about $\lambda_j = \lambda_j \pm 5$ where $\lambda_j = L_j/h_2$. The data of Press (1961) for M = nh_1/b = 0.0625, 0.125 and 0.25 were analysed by Silvester (1964) and Wanoschek and Hager (1989) to propose

$$\frac{L_j}{h_2-h_1} = 7.1(1+10M). \quad (2)$$

Ohtsu (1976) correlated the length of hydraulic jump L_j by the relation

$$\log_{10}\left(\frac{L_j}{\Delta H}\right) = 1.71\zeta + 0.315n + 1.58. \quad (3)$$

where $\zeta = \Delta H/H_i$ is the relative energy loss across the jump, ΔH is the energy loss in the jump and H_i is the total upstream head. In the rectangular channel, various expressions for the jump length L_j proposed by earlier workers have been summarized by Elevatorski (1959) and Rajaratnam (1967).

The hydraulic jump is analogous to a shock wave (Duncan et al. 1967). In both situations, the upstream and downstream flows approach to certain limiting conditions: in shock waves these are the well known Rankine-Hugoniot relations (Thompson 1976) whereas for hydraulic jump these are the Belanger relations for a rectangular channel. The shock wave structure in laminar flow, reported by Germain and Guiraud (1962) and Thompson(1976), show that the molecular viscous normal stress plays the dominant role. The measurement of Resch et al. (1976) in a turbulent hydraulic jump were also analyzed in depth average flow and demonstrate that Reynolds normal stresses contribute significantly to axial momentum balance. This is also supported by the data of Rouse et al. (1959), Xin (1993) and Liu and Drewes (1994).

The objective of the present work is to analyze the flow structure of a turbulent hydraulic jump in a channel of arbitrary cross-section based on depth averaged Reynolds equations of mean turbulent flow at large Reynolds numbers. The general equation for the structure of the hydraulic jump in channels of arbitrary cross-sections has been proposed. The closure model for averaged normal Reynolds stress is expressed as a product of the constant eddy viscosity and the gradient of the depth averaged axial velocity with respect to the axial distance. The solution for trapezoidal channels (that includes rectangular and triangular channels as special cases) has been obtained for the surface profile of the hydraulic jump. The length scale of roller L_R is of the order of overall length scale L_j of the hydraulic jump. The experimental data on the roller length reported by Hager and Bremen (1989) and Hager et al. (1990) for $2 < F_1 < 15$ in the rectangular channel and Rajaratnam (1964) in the triangular channel compare well with the present length prediction for the roller.

2. Structure of hydraulic jump

In the two dimensional mean turbulent flow the Reynolds momentum equation in the axial direction and the equation of continuity have been averaged over the channel depth of arbitrary cross-section, as described in Appendix-A to obtain the following expressions

$$\frac{\partial}{\partial x}\left[(1+\beta)AU^2 + \int \frac{p}{\rho}\,dA - \frac{1}{\rho}A\,T_{xx}\right] = -\frac{1}{\rho}\tau_w$$
$$\frac{\partial}{\partial x}(A\,U) = 0. \quad (4a)$$

The Reynolds normal momentum equation, following Rajaratnam (1967) and Madsen and Svendsen (1983), may be approximated by hydrostatic pressure distribution as

$$p = \rho g(h-z). \quad (4b)$$

The various variables in the equations (4) denote

A = Area of arbitrary channel cross-section

$F_1^2 = (Q^2/gA_1^3)(dA_1/dh_1)$, upstream Froude number

g = Gravitational acceleration

h = Depth of fluid layer above the bottom

$k = \frac{1}{A}\int\left(1-\frac{z}{h}\right)dA$, geometrical factor

Q = discharge of fluid and ρ is fluid density

$S = \frac{1}{\rho g}\frac{1}{A}\int pdA = kh.$, hydrostatic depth averaged pressure per unit specific weight

$U = \frac{1}{A}\int udA.$, cross-sectional averaged velocity in x-direction

$T_{xx} = \frac{1}{A}\int -\rho\overline{u'u'}dA.$, depth averaged Reynolds normal stress in the flow direction.

x = Streamwise horizontal co-ordinate of the flow
z = Vertical co-ordinate above the bottom of the channel
$\alpha = h_1/h_2$, the sequent depth ratio
β = kinetic energy correction factor
τ_w = frictional force per unit flow depth at the bottom
Subscripts 1 and 2 denote values upstream and downstream of the hydraulic jump.

The relations (4), after neglecting skin friction along the bottom and assuming a kinetic energy correction factor $\beta = 1$ for the uniform flow may be integrated to obtain

$$UA = U_1A_1 = U_2A_2 = Q \tag{5}$$

$$\begin{aligned} A(U^2 + gS - T_{xx}/\rho) &= A_1(U_1^2 + gS_1) \\ &= A_2(U_2^2 + gS_2) \end{aligned} \tag{6}$$

The upstream Froude number may be defined as

$$\begin{aligned} F_1^2 &= \frac{U_1^2}{gD_1} = \frac{Q^2}{gA_1^3}\frac{dA_1}{dh_1}. \quad \hat{F}_1^2 = \frac{U_1^2}{gS_1} = C_1F_1^2. \\ D_1 &= C_1S_1 = A_1\left(\frac{dA_1}{dh_1}\right)^{-1} \end{aligned} \tag{7}$$

The above equations are expressed in terms of following non-dimensional variables

$$\phi(x) = \frac{A(x)}{A_1}. \quad \lambda = H(x)\frac{k}{k_1}. \quad H(x) = \frac{h(x)}{h_1}. \tag{8}$$

The analysis of equations (5) and (6) for two flow invariants yields the condition for the formation of the hydraulic jump as

$$\phi_2(\lambda_2\phi_2 - 1) = C_1F_1^2(\phi_2 - 1) \tag{9}$$

$$\phi_2 = \frac{A_2}{A_1}. \quad \lambda_2 = H_2\frac{k_2}{k_1}. \quad H_2 = \frac{h_2}{h_1} = a^{-1}. \tag{10}$$

The depth-averaged Reynolds normal stress is given by

$$\begin{aligned} &T_{xx}\phi^2\phi_2(\lambda_2\phi_2 - 1)/U_1^2 = -\phi_2(\lambda_2\phi_2 - 1)(\phi - 1) \\ &+\phi(\lambda\phi - 1)(\phi_2 - 1). \end{aligned} \tag{11}$$

The closure model in the jump, described in Appendix-B, where depth averaged normal Reynolds stress has been expressed as product of the eddy viscosity and gradient of the depth-averaged axial velocity with respect to streamwise x-direction

$$T_{xx} = \rho\nu_\tau\frac{\partial U}{\partial x}. \tag{12}$$

The eddy viscosity ν_τ based on the overall velocity scale $\Delta U = U_1 - U_2$ and the length scale $\Delta h = h_2 - h_1$ of the hydraulic jump is adopted as

$$\nu_\tau = \in (U_1 - U_2)(h_2 - h_1). \tag{13}$$

where ε is a universal constant independent of channel geometry. Based on relations (8) and (9), equation (11) describing the structure of turbulent hydraulic jump in the direction of flow is given by

$$\begin{aligned} &\in (\lambda_2\phi_2 - 1)(\phi_2 - 1)(H_2 - 1)h_1\frac{\partial\phi}{\partial x} = \\ &-\phi_2(\lambda_2\phi_2 - 1)(\phi - 1) + \phi(\lambda\phi - 1)(\phi_2 - 1). \end{aligned} \tag{14}$$

subjected to the following upstream and downstream boundary conditions

$$\begin{aligned} &x \to -\infty.\ \phi \to 1.\ \lambda \to 1: \\ &x \to +\infty.\ \phi \to \phi_2, \lambda \to \lambda_2. \end{aligned} \tag{15}$$

3. Trapezoidal channel

In the trapezoidal channel of bottom width b, side slope n and upstream depth of flow h_1 the non-dimensional variables are

$$\begin{aligned} &\phi = \frac{MH+1}{M+1}H. \quad \lambda\phi = \frac{2MH+3}{2M+3}H^2. \\ &M = \frac{nh_1}{b}. \quad C_1 = \frac{6(M+1)^2}{(2M+1)(2M+3)}. \end{aligned} \tag{16}$$

The necessary condition (9) for the formation of the hydraulic jump reduces to

$$\begin{aligned} &\left[(2MH_2 + 3)H_2^2 - (2M+3)\right]\frac{2M+1}{6(M+1)^2} \\ &= F_1^2\left[1 - \frac{M+1}{H_2(MH_2+1)}\right] \end{aligned} \tag{17}$$

The solution $\alpha = 1/H_2$ of the jump equation (17) for various values of M against the upstream Froude number F_1 has been described by Silvester (1964), Ali and Ridgway (1977), Wanoschek and Hager (1989) and Hager (1992).

The flow structure of hydraulic jumps governed by equation (14) subjected to boundary conditions (15) may formally be expressed in terms of the non-dimensional upper surface profile $\eta(X)$ defined by

$$\eta = \frac{h - h_1}{h_2 - h_1}. \quad X = \frac{x}{h_2} \tag{18}$$

In the trapezoidal channel, the equations (14) and (15) yield after simplification

$$\begin{aligned} &\in (1-\alpha)K_1K_2(2M\psi + \alpha)\frac{\partial\eta}{\partial X} \\ &= \eta(1-\eta)\left[f(\psi) \mid +B\right] \end{aligned} \tag{19}$$

$$X \to -\infty. \quad \eta \to 0: \quad X \to +\infty. \quad \eta \to 1 \tag{20}$$

Here functions $\psi, f(\psi)$ and $G(\psi)$ are

$$\psi = \alpha + \eta(1-\alpha). \tag{21a}$$

$$f(\psi) = K_1\{2M^2\psi^3/3 + [2M^2(1+\alpha) + 5M\alpha]\psi^2 +$$

$$[2M^2(1+\alpha+\alpha^2) + 5M\alpha(1+\alpha) + 3\alpha^2]\psi\} \tag{21b}$$

$$G(\psi) = \frac{\partial f(\psi)}{\partial \psi}$$

$$= K_1 \begin{Bmatrix} 2M^2\psi^2 + 2\left[2M^2(1+\alpha) + 5M\alpha\right]\psi \\ +2M^2(1+\alpha+\alpha^2) + 5M\alpha(1+\alpha) + 3\alpha^2 \end{Bmatrix} \tag{21c}$$

Further, the geometric constants are

$$K_1 = M(1+\alpha) + \alpha. \quad K_2 = 2M(1+\alpha+\alpha^2) + 3\alpha(1+\alpha). \quad B = \alpha(M+1)(M+\alpha)K_2. \tag{22}$$

The asymptotic values $M = 0$ and $M^{-1} = 0$ correspond to rectangular and triangular channels, respectively. The solution of the momentum equation (19) subject to the boundary conditions (20) for fixed value of M in the range $0 \le M \le \infty$ predicts the depth profile $\eta(X)$ and the length of hydraulic jump L_j. However, axial length of hydraulic jump may also be based on maximum slope of the profile, in analogy with shock wave thickness (Thompson 1976), as

$$L_j = (h_2 - h_1) / \left(\frac{\partial h}{\partial x}\right)_m \tag{23}$$

where $\left(\frac{\partial h}{\partial x}\right)m$ is the slope of water surface profile for the mean depth $h_m = (h_2+h_1)/2$. Based on equation (19), the length of jump (23) may be expressed as

$$L_j / h_2 = \in (1-\alpha)\Delta \tag{24}$$

$$\Delta = \frac{4K_1^2 K_2}{f(\psi_m) + B}. \quad \psi_m = \frac{1}{2}(1+\alpha) \tag{25}$$

$$f(\psi_m) + B = \begin{bmatrix} (7 + 32\alpha + 41\alpha^2 + 32\alpha^3 + 7\alpha^4) \\ M^3 + 12\alpha(1+\alpha)^3 M^2 + \alpha^2 \\ (41 + 74\alpha + 41\alpha^2)M + 18\alpha^3(1+\alpha) \end{bmatrix} / 4. \tag{26}$$

where K_1, K_2 and B are given by relations (22). In rectangular and triangular channels the hydraulic jump function Δ is simplified to

$$\Delta = \frac{8}{3}. \quad M = 0. \tag{27}$$

$$\Delta = \frac{32(1+\alpha)^2(1+\alpha+\alpha^2)}{7 + 32\alpha + 41\alpha^2 + 32\alpha^3 + 7\alpha^4}. \quad M^{-1} = 0. \tag{28}$$

The solution of the upper surface depth distribution in the hydraulic jump governed by equation (19) subject to the upstream and downstream boundary conditions (20) may be expressed as

$$\eta^{\beta_1}(1-\eta)^{-\beta_2}\left[f(\psi) + B\right]^{-\beta_3} = \exp\left[\frac{\beta_1\beta_2 X}{\in(1-\alpha)} + L\right] \tag{29}$$

where L is an unknown additive constant. Further, β_1, β_2 and β_3 are given by relations

$$\beta_1 = \frac{f(1) + B}{K_1 K_2 (2M + \alpha)}. \quad \beta_2 = \frac{f(\alpha) + B}{K_1 K_2 (2M + 1)\alpha} \tag{30a,b}$$

$$\beta_3 G(\psi_m) = 4K_1^2 K_2 \beta_1 \beta_2 - 2(\beta_1 + \beta_2)(f(\psi_m) + B). \tag{30c}$$

where $G(\psi_m)$ at $\psi = \psi_m$ is given by relation (21c)

3. Results and discussion

The hydraulic jump length L_j/h_2 predicted by expression (24) involves the function Δ which depends on parameter M for the trapezoidal channel and the upstream Froude number F_1. In a rectangular channel, the non-dimensional length L_j/h_2 is displayed in Fig 2 using experimental data of Bakhmeteff and Matzke (1936), Moore (1943), USBR (1958), Rouse et al. (1959), and Sananes & Fortey (1966). The relations proposed by Silvester (1964)

$$\frac{L_j}{h_1} = 9.75(F_1 - 1)^{1.01} \tag{31}$$

and Rajaratnam and Subramanya (1968)

$$\frac{L_j}{h_1} = 5.08F_1 - 7.82 \tag{32}$$

are also shown in Fig 2. In rectangular channels with M = 0, Δ = 8/3 and the present jump relation (24) to the non-dimensional length L_j/h_2 against the upstream Froude number F_1 is shown in Fig 2. The proposed universal constant is determined by fitting relation (24) with the data as

$$\frac{L_j}{h_2} = 6.9(1-\alpha). \quad \varepsilon = 2.578. \tag{33}$$

which shows that for large Froude numbers $F_1 \to \infty$ the jump length L_j/h_2 is independent of the upstream Froude number. As suggested by Subramanya (1982) for $F_1 > 5$, the jump length L_j/h_2 = 6.1 is practically constant, whereas Elevatorski (1959) proposed the constant 6.9, Further, the measurements for very large F_1 show a slight decrease of L_j/h_2 when compared with theory

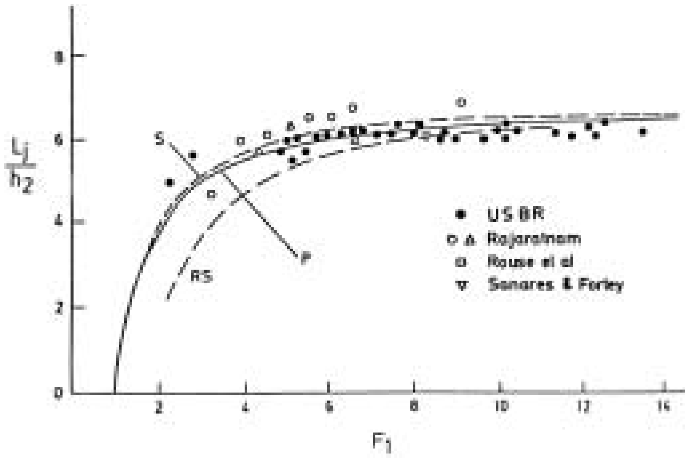

Fig. 2 Comparison of length of hydraulic jump L_j/h_2 in a rectangular open channel with experimental data and expressions proposed by previous workers.

S : Silvester: $\frac{L_j}{h_1} = 9.75(F_1 - 1)^{1.01}$,

RS: Rajaratnam and Subramanya: $\frac{L_j}{h_1} = 5.08F_1 - 7.82$,

P : Present relation (33): $\frac{L_j}{h_2} = 6.9(1-\alpha). \quad \varepsilon = 2.578.$. \\

(Rangaraju 1993). There is a need for better data at higher Froude numbers.
The universality of the numerical constant ε = 2.578 determined from experimental data for the rectangular channel implies independence of channel geometry. For a trapezoidal channel, the length relation (24) L_j/h_2 for fixed M = 0.1 to 0.4 compared with experimental data of Wanoschek and Hager (1989) is very encouraging (Fig 3). In the triangular channel $M^{-1} = 0$, the comparison of the present relation (28) with data of Argyropoulous (1961) as displayed in Fig 4 is also good.
The quantity L in the solution (29) for the surface profile includes physical uncertainty in the location of the axial origin of the hydraulic jump, analogous to that in the shock wave (Thompson 1976). The profile may be shifted to any desired streamwise location in the x-direction for comparison of a particular data set. Let us define non-dimensional streamwise distance

$$\xi = (X - X_1)/(X_2 - X_1) \tag{34}$$

where $X = X_1$ and $X = X_2$ are the streamwise locations of the toe and the end of the hydraulic jump. The closed form solution of the jump profile (29) may also be expressed as

$$In\ J(\eta) = \gamma X + L. \quad \in \gamma = \beta_1\beta_2/(1-\alpha) \tag{35}$$

$$J(\eta) = \eta^{\beta_1}(1-\eta)^{-\beta_2}\left[f(\psi)+B\right]^{-\beta_3} \tag{36}$$

The solution (35) becomes independent of the constant L, if the following boundary conditions on depth profile, cut off at the toe and end of jump, are adopted

$$\xi = 0. \quad \eta = \eta_1 : \quad \xi = 1. \quad \eta = \eta_2. \tag{37}$$

The relations (35) and (37) lead to the following relations to cut off the depth profile

$$X_1\gamma + L = In\ J(\eta_1). \quad X_2\gamma + L = In\ J(\eta_2). \tag{38}$$

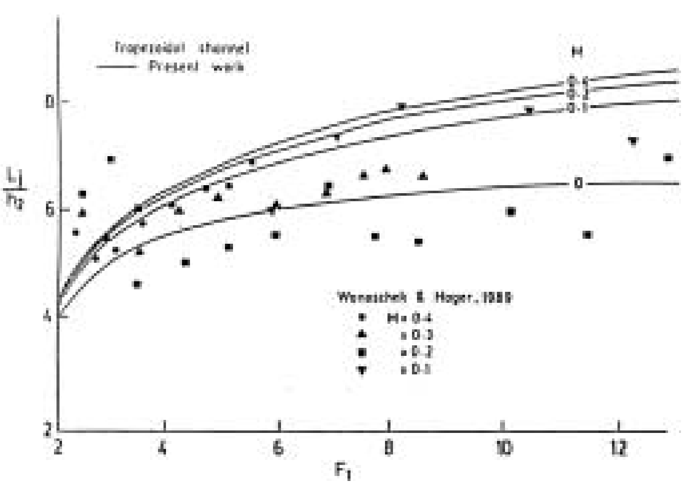

Fig. 3 Comparison of hydraulic jump length Lj/h_2 from present relation (24) with experimental data of Wanoschek and Hager (1989) and Hager (1992) for trapezoidal channel with M = 0.1 - 0.4.

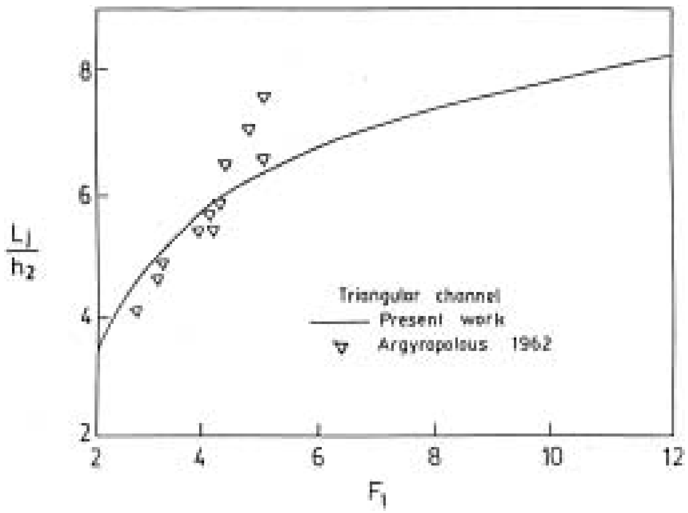

Fig. 4 The comparison of hydraulic jump length Lj/h_2 from present relation (24) and experimental data of Argyropoulous (1961) for triangular channel.

The solution (35) under boundary conditions (37) and (38) becomes independent of the streamwise constant L, which may be expressed as

$$\xi = \frac{In\ J(\eta) - In\ J(\eta_1)}{In\ J(\eta_2) - In\ J(\eta_1)}. \tag{39}$$

Here η_1 and η_2 are the cut off values of axial flow depth levels at the toe and at the end of the jump. The solution may be estimated for 5 percent cut off. As the surface profile is very flat towards the end of the jump, large personal errors are introduced in the determination of jump length L_j and η_2. Further, at the toe the point of beginning of the hydraulic jump is not precisely defined (Rajaratnam 1967), and therefore the value of η_1 is uncertain. The data from various sources as considered by Rajaratnam and Subramanya (1968) show that $\eta_1 = 0.15$ and $\eta_2 = 0.95$ may be appropriate.
In a rectangular channel, the upper surface profiles (38) for F_1 = 2,5 and ∞ are displayed in Fig.5, which compare well with the data from Bakhmeteff and Matzke (1936), Moore (1943) and Rouse et. al (1958). The data shows that for $F_1 > 5$ the deviation in the jump profile is small. Therefore, for large Froude numbers, the hydraulic jump profile approaches a universal solution provided the variables are appropriately non-dimensionalized. For large values $F_1 \geq F_1^*$. , the solution (24) of hydraulic jump length L_j/h_2 and solution (38) of hydraulic jump structure η in the trapezoidal channel (for M fixed) is independent of the upstream Froude numbers F_1 approach a limiting form (depending on M). The effect of air-water interaction producing the roller near the surface, has been displayed in Fig 1, whose transverse length scale δ_y was suppressed in the depth averaged integral analysis. Let longitudinal length scale of the roller due to air-water interaction be $\delta_x = L_R$. The present analysis based on depth averaged Reynolds equations of mean turbulent flow in a channel of arbitrary cross-section has been sufficiently general, and it is reasonable to postulate that the length scale of rollers L_R is of the order of the overall length scale L_j of the hydraulic jump

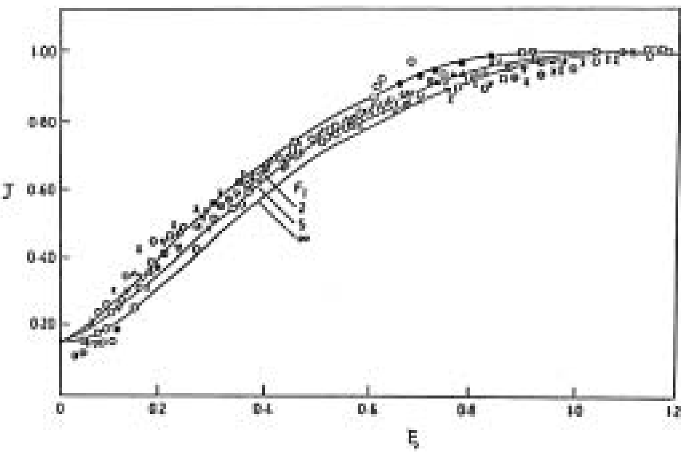

Fig. 5 Comparison of water depth free surface profile (39) for hydraulic jump in rectangular channel with the data. Bakhmeteff and Matzke: $F_1 = 8.63$ ○; 5.53 ●; 4.09 ▲; 2.92 □; 1.98 ▼; Moore: $F_1 = 6.7$ ▽; 4.95 ⊕; 2.24 ⊖; 4.05 ∗. Rouse, Siao and Nagaratnam $F_1 = 4$ ×.

$$L_j = mL_R \tag{40}$$

where m is again a universal constant. In a rectangular channel, for $F_1 \geq 4$, Sarma and Newnham (1973) displayed the similarity of surface profiles and proposed $L_j / L_R = m = 1.3$. Further, different workers adopted various other variables while non-dimensionalizing the streamwise distance x, that also support the order hypothesis (40). For example, Schroder (1963) and Sarma and Newnham (1973) adopted L_R, Gupta (1967) L_j, Rajaratnam and Subramanya (1968) the value of X where 0.75 (h_2 -h_1), and Bakhmeteff and Matzke (1936) h_2 -h_1.

In a rectangular channel, the experimental data for the roller length L_R have been summarized in Table 1 of Hager et. al (1990). Sarma and Newnham (1973) proposed

$$\frac{L_R}{h_1} = 6.73(F_1 - 1). \quad F_1 \leq 4 \tag{41a}$$

whereas Hager et. al (1990) proposed

$$\frac{L_R}{h_1} = 8(F_1 - \frac{3}{2}). \quad 2.5 \leq F_1 \leq 8 \tag{41b}$$

The non-dimensional form as proposed by Busch (1981) is

$$\frac{L_R}{h_1} = f(F_1 . \omega . R_1) \tag{42}$$

where L_R depends on the initial flow depth to width ratio $\omega = h_1/b$ whereas the influence of inflow Reynolds number for R_l (where for rectangular channel $R_1 = Q(b\nu)^{-1}$) was found weak. Hager et. al (1990) also expressed their data by the following empirical relations

$$\frac{L_R}{h_1} = -12 + 160 \tanh(F_1 / 20). \quad \omega < 0.1 \tag{42a}$$

$$\frac{L_R}{h_1} = -12 + 100 \tanh(F_1 / 12.5). \quad 0.1 < \omega < 0.7 \tag{42b}$$

In present work, the roller length L_R may be predicted by utilizing the order hypothesis (40), along with fundamental streamwise length scale L_j relation (24) for the hydraulic jump based on the integral momentum equation as

$$\frac{L_R}{h_2} = \epsilon_R (1 - \alpha)\Delta : \quad \epsilon = m \epsilon_R \tag{43}$$

Here Δ for trapezoidal channel is given by relations (25)-(28). The experimental data for roller length reported by Hager and Bremen (1989) and Hager et al. (1990) for $2 < F_1 < 15$ for rectangular channel have been compared with the present relation (43) to determine the universal constants ε_R and the order constant m from relation (40) as

$$\epsilon_R = 1.95 \quad m = 1.254. \tag{44}$$

Consequently, for rectangular channel the relation (43) becomes

$$\frac{L_R}{h_2} = 5.2(1 - \alpha). \tag{45}$$

The comparison of data shown in Fig 6 supports prediction (45) for the roller length provided $F_1 \geq 4$. However, for $2 \leq F_1 < 4$ the data are slightly under-predicted and need further consideration. The universal numerical constants (44) determined from rectangular channel data may be tested for independence of the channel geometry. In the triangular channel $M^{-1} = 0$, the roller length (43) is predicted by

$$\frac{L_R}{h_2} = \epsilon_R (1 - \alpha)\Delta : \tag{46}$$

where Δ is given by (28). The experimental data on the roller length L_R in the triangular channel was reported by Rajaratnam (1964) for $F_1 \leq 7$ as displayed in Fig 7. This compares well with prediction (46). In order to further test the universal values (44), the data for rollers length L_R in trapezoidal channel is needed.

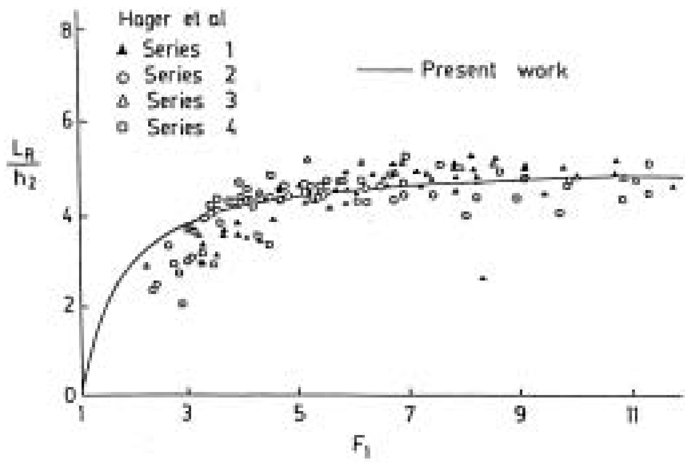

Fig. 6 Rectangular Channel: Comparison of the present prediction -----
$$\frac{L_R}{h_2} = 5.2(1 - \alpha).$$
with experimental data of roller length in the hydraulic jump of rectangular channel. Hager et. al (1989): ▲ Series 1; ○ Series 2: Hager et. al (1990): Δ Series 3; □ Series 4.

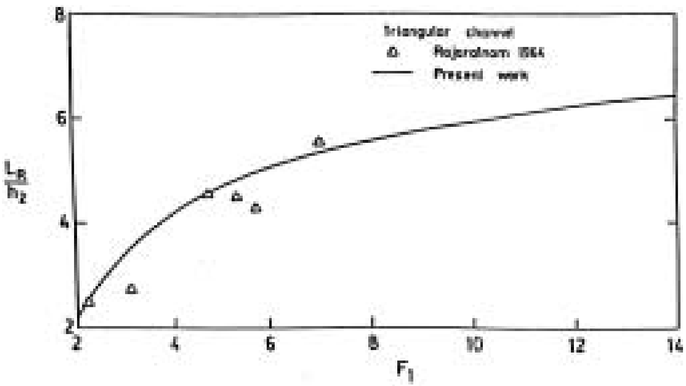

Fig. 7 Comparison of (46) for L_R/h_2 with experimental data of Rajaratnam (1964) for roller length of the hydraulic jump in the triangular channel.

4. Conclusions

1) The hydraulic jump in open channel flow is analogous to a shock wave in a compressible fluid in several respects, such as the role of parameter (Froude/Mach number), flow invariants jump conditions, passive role of bottom wall and significance of normal stress in jump structure profile.
2) The Reynolds equations of two-dimensional mean turbulent motion in a channel for depth averaged flow have been analysed at large Reynolds numbers and to the lowest order inertia, hydrostatic pressure and Reynolds normal stress terms play predominant roles.
3) A sufficiently general equation for the turbulent hydraulic jump profile in an open channel of arbitrary cross section has been proposed. The equation has been closed by a simple eddy viscosity model involving a constant ε, where the passive role of the bottom bed imposing the shear stress has been neglected (Clauser 1956). Further, extensive data supports constant ε = 2.578 a universal number.
4) The theory shows that at large upstream Froude number ($F_1 \to \infty$) the hydraulic jump length L_j/h_2 is independent of F_1 in channels of arbitrary cross sections.
5) The closed form expressions for length of hydraulic L_j proposed for trapezoidal channels includes rectangular and triangular channels as extreme cases. The predictions for L_j/h_2 agree with extensive data in the literature for $F_1 > 3$.
6) The postulate that length scale of rollers L_R is of the order of the overall length scale L_j of the hydraulic jump, (i.e. $L_j = m\,L_R$ where m is a universal constant) has been proposed here. It is supported by extensive data on the roller with an universal constant m = 1.3.

Appendix A: Analysis for Reynolds equations in a hydraulic jump

The Reynolds equations of mean turbulent motion in the two dimensional flow of an incompressible fluid subjected to pressure gradient are

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{A1}$$

$$\begin{aligned}&\rho\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{x}}{\partial x}\\&+\frac{\partial \tau_{xz}}{\partial z}+\rho\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)\end{aligned} \tag{A2}$$

$$\begin{aligned}&\rho\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=\rho g-\frac{\partial p}{\partial z}+\frac{\partial \tau_{x}}{\partial x}\\&+\frac{\partial \tau_{zz}}{\partial z}+\rho\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)\end{aligned} \tag{A3}$$

Here u is the streamwise velocity in x-direction and w is the normal velocity in z-direction, p is the static pressure, ρ is fluid density and g is the gravitational acceleration. Further, τ_{ij} is the Reynolds stress tensor defined by

$$\begin{aligned}&\tau_{xx}=-\rho\overline{u'u'}.\quad \tau_{xz}=\tau_{zx}=-\rho\overline{u'w'}.\\&\tau_{zz}=-\rho\overline{w'w'}\end{aligned} \tag{A4}$$

The boundary conditions at bottom z = o and free surface of fluid z = h(x) are

$$z=0.\quad u=w=0.\quad \tau_{ij}=0. \tag{A5a}$$

$$z=h(x).\quad u=U_h(x).\quad \tau_{ij}\neq 0 \tag{A5b}$$

Multiplying the continuity equation (A1) by u and adding it to the momentum equation (A2) we get

$$\begin{aligned}&\rho\left(\frac{\partial}{\partial x}uu+\frac{\partial}{\partial z}uw\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\\&+\frac{\partial \tau_{xz}}{\partial z}+\rho\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)\end{aligned} \tag{A6}$$

Equation (A6) after separation of terms in streamwise and normal derivatives become

$$\begin{aligned}&\frac{\partial}{\partial x}\left(u^2+\frac{p}{\rho}-\frac{1}{\rho}\tau_{xx}-\nu\frac{\partial u}{\partial x}\right)\\&=\frac{\partial}{\partial z}\left(-uw-\frac{1}{\rho}\tau_{xz}+\nu\frac{\partial u}{\partial z}\right)\end{aligned} \tag{A7}$$

We consider the integral approach based on depth averaged equations of motion.The continuity equation (A1) and axial momentum equation (A7) are integrated across the fluid layer area A of the arbitrary cross-section to obtain

$$\frac{\partial}{\partial x}\int u\,dA=0 \tag{A8}$$

$$\frac{\partial}{\partial x}\int\left(u^2+\frac{p}{\rho}-\frac{1}{\rho}\tau_{xx}-\nu\frac{\partial u}{\partial x}\right)dA=-\frac{1}{\rho}\tau_w \tag{A9}$$

where τ_w is the frictional force per unit flow depth at the bottom surface. The normal momentum equation (A3) may also be averaged over the depth of the flow. From the analysis of experimental data, however, Rajaratnam (1967) concluded that for all values of x, the static pressure p may be approximated by hydrostatic pressure distribution.

$$p = \rho g(h - z) \tag{A10}$$

The deviations of pressure from the hydrostatic value would have little effect on the momentum balance in the streamwise direction (see also Appendix A of Madsen and Svendsen 1983). For a turbulent hydraulic jump, the molecular viscous stresses in the fluid are ignored and equations (A8)-(A9) become

$$\frac{\partial}{\partial x}\left[(1+\beta)A\,U^2 + \int \frac{p}{\rho}dA - \frac{A}{\rho}\boldsymbol{T}_{xx}\right] = -\frac{1}{\rho}\tau_w$$
$$\frac{\partial}{\partial x}(A\,U) = 0 \tag{A11}$$

Here U(x) average velocity over the channel of arbitrary cross section, T_{xx} the depth averaged Reynolds normal stress in direction of flow and β energy correction factor are given by

$$U = \frac{1}{A}\int u\,dA, \quad \boldsymbol{T}_{xx} = \frac{1}{A}\int -\rho\overline{u'u'}\,dA,$$
$$\beta = \frac{1}{A}\int\left(\frac{u}{U}\right)^2 dA - 1. \tag{A12}$$

For uniform flow in a channel $\beta = 0$. The hydraulic jump invariants may be determined from the relations (A10) and (A11) from the boundary conditions $x \to \pm\infty, T_{xx} \to 0$. The open momentum equation (A11) in hydraulic jump profile, may be closed by a closure hypothesis.

Appendix B: Closure model for Reynolds normal stress

The eddy viscosity may be defined as Reynolds stress per unit strain rate of the fluid motion. More general expressions, for example in *k*-ε method are described by Long at al (1990) and Liu & Drewes (1994). The expression for the strain rate depends on the role of the bed at the bottom surface in formation and development of hydraulic jumps, and its analogy with shock wave in a compressible fluid are considered below.

1) The bottom surface over which the hydraulic jump (or shock wave) is formed does not play a predominant role, except that the skin friction at the bottom wall affects the physical location of the hydraulic jump (or shock wave). The flow in the hydraulic jump above the bottom surface has been regarded as a turbulent shear layer having an air-water interaction on the upper surface forming the rollers in the mixing layer, the extent of which depends on the magnitude of the upstream Froude number.

2) The outer portion of the turbulent boundary layer, away from wall the flow behaves like a wake and the eddy viscosity defined by Clauser (1956) on appropriate outer scales is constant. In a hydraulic jump, the passive role of bottom wall is neglected and the eddy viscosity based on appropriate scales should also be a universal number, in accordance with the work of Clauser (1956) and Afzal(1996).

3) The present work regards the hydraulic jump analogous to a shock wave (Duncan et al. 1967) where depth averaged momentum equations in the jump contain Reynolds normal stress terms. The shock wave structure in laminar flow is reported by Germain and Guiraud (1962) and Thompson (1976), where molecular normal viscous stress plays the dominant role. The measurements of Rouse et al. (1959), Resch et al (1976), Xin (1993) and Liu and Drewes (1994) in a turbulent hydraulic jump demonstrate that Reynolds normal stresses contribute significantly to axial momentum balance. Rouse et al (1959) adopted the Boussinesq eddy viscosity concept for Reynolds shear stress only.

4) In a boundary layer flow, the gradients in normal z-direction are dominant, whereas in a shock wave gradients in streamwise x-direction are dominant (Thompson 1976). In a hydraulic jump, the eddy viscosity may be defined as ratio depth between averaged Reynolds stress to the strain rate of depth averaged velocity in streamwise direction.

In view of these facts, the closure model for depth averaged normal Reynolds stress in axial direction of hydraulic jump may be defined as the product of the eddy viscosity and gradient in x-direction depth-averaged velocity U(x) given by

$$\frac{1}{A}\int -\rho\overline{u'u'}\,dA = T_{xx} = \rho\upsilon_\tau\frac{\partial U}{\partial x}. \tag{B1}$$

Consequently, the eddy viscosity ν_τ based on the overall velocity scale $\Delta U = U_1 - U_2$ and the length scale $\Delta h = h_2 - h_1$ of the hydraulic jump is adopted as

$$\nu_\tau = \in (U_1 - U_2)(h_2 - h_1). \tag{B2}$$

Here ε is a universal constant, independent of channel geometry, following Clauser (1956), because the passive role of bottom wall on the jump has been ignored. The passive role re-appears in the jump profile solution (35) as an additive constant L, which in analogy with shock wave in compressible fluid (Thompson 1976), represents uncertainty in the geometric location of jump in x- direction.

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Notations

$A(x)$ area of channel of arbitrary cross-section based on flow depth.

b bottom width of the trapezoidal channel.

B constant defined by relation (22).

C_1, D_1 channel geometry constants (7) for intial flow depth h_1.

$f(\psi)$ function defined by relation (21b).

$f(\psi_m)$ function (26) at mean depth h_m

F_1 upstream Froude number.

$F_1^2 = (Q^2 / gA_1^3)(dA_1 / dh_1)$ function defining F_1.

g gravitational acceleration.

$h(x)$ depth of fluid layer above the bottom.

$h_m = (h_1 + h_2)/2$ mean of upstream and downstream depths in a hydraulic jump.

$H(x) = h/h_1$ non-dimensional depth based on upstream depth of flow.

$H_2 = h_2/h_1 = \alpha^{-1}$ sequent depth in hydraulic jump.

$J(\eta)$ depth averaged profile function(36).

$k = \frac{1}{A}\int\left(1-\frac{z}{h}\right)dA$ channel geometrical factor, a function of flow depth.

K_1, K_2 constants in jump profile equation (19).

L arbitrary constant representing the streamwise location of the jump origin.
L_j length of the hydraulic jump.
L_R roller length in the formation of the hydraulic jump.
$m = L_j/L_R$ the constant of order hypothesis (40).
$M = nh_1/b$ side wall constant in trapezoidal channel.
n side wall slope of the trapezoidal channel.
p hydrostatic pressure distribution.
Q discharge in channel flow.
$R_1 = U_1 h_1/\nu$ Reynolds number of in flow to the jump.
$S = \frac{1}{\rho g}\frac{1}{A}\int pdA = kh.$ hydrostatic depth based on averaged pressure per unit specific weight.
u(x,z) local velocity at a point in streamwise x-direction.
$U = \frac{1}{A}\int udA.$ cross-sectional averaged velocity in x-direction.
$U_h(x)$ free surface velocity at z = h(x).
w(x,z) local velocity at a point in normal z-direction.
$T_{xx} = \frac{1}{A}\int -\rho\overline{u'u'}dA.$ depth averaged Reynolds normal stress in the flow direction.
x streamwise horizontal co-ordinate of the flow.
$X = x/h_2$ non-dimensional streamwise variable.
z vertical co-ordinate measured above the bottom wall.

Greek Symbols

$\alpha = h_1/h_2$, the sequent depth ratio.
β kinetic energy correction factor.
β_1, β_2, β_3 constants of integration defined by relations (30).
$\gamma = \beta_1\beta_2/[\varepsilon(1-\alpha)]$ channel constant function of M.
ε eddy viscosity (13) universal contant independent of channel geometry.
ε_R universal number independent of channel geometry in roller length. relation (43)
ϕ area of flow non-dimensional by upstream flow area.
$\psi = \alpha + \eta\ (1-\alpha)$ non-dimensional flow depth based on down stream flow depth.
$\psi_m = (1 + \alpha)/2$ non-dimensional function at $h=h_m$ mean depth.
$\eta(x) = [h(x) - h_1]/(h_2 - h_1)$, non-dimensional flow depth.
λ jump profile function (8).
Δ jump length function (24).
ν molecular kinematic viscosity of fluid.
ν_τ eddy viscosity of flow in the jump.
ρ fluid density.
τ_{xx}, τ_{zz} Reynolds normal stresses.
τ_{xz} Reynolds shear stress.
τ_w frictional force per unit flow depth at the bottom.
ξ non-dimensional axial co-ordinate (34) of flow in the jump.

{Subscripts}
1 upstream of jump.
2 downstream of jump.