Hydraulic geometry of straight alluvial channels and the principle of least action

A propos de l'efficacité des rivières dans leur régime et du principe de moindre action

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ABSTRACT
Natural rivers exhibit regular hydraulic geometry relationships for which no widely accepted explanation has been given. This paper applies the physical principle of least action to the determination of stable alluvial-channel form. For steady, uniform alluvial-channel flow, both theoretical inferences and a case study show that least action occurs when the criteria of minimum potential energy and MFE (Maximum Flow Efficiency, defined here as the maximum sediment transporting capacity per unit available stream power) are satisfied. The consistency between bankfull hydraulic geometry relationships of natural channels and those of maximally efficient or 'least action' channels identified in this study demonstrates that alluvial channels commonly adjust to a maximally efficient section. Support for the use of the extremal hypotheses of maximum sediment transporting capacity and minimum stream power is provided by illustrating that they are essentially expressions of, and hence subsumed by, the more general principle of MFE.

RÉSUMÉ
Les cours d'eau naturels présentent des régulières relations de géométrie hydraulique, pour lesquelles il n'y a pas d'explication largement admise. Cet article applique le principe de moindre action à la détermination de la forme des chenaux alluviaux. Pour l'écoulement permanent et uniforme en canal alluvial, une analyse théorique ainsi qu'une étude de cas montrent que la moindre action est présente lorsque le critère d'énergie potentielle minimale et de MFE (Efficacité Maximale de l'Ecoulement, définie ici comme la capacité maximale de transport sédimentaire par unité de puissance du courant total) sont vérifiés. La cohérence entre les relations de géométrie hydraulique des chenaux naturels et celles des chenaux d'efficacité maximale ou « de moindre action » identifiés dans cette étude, montre que les chenaux alluviaux s'adaptent généralement à une section d'efficacité maximale. L'usage des hypothèses sur les extremums : maximum de la capacité de transport sédimentaire et minimum de puissance du courant total, est étayé en soulignant qu'elles sont essentiellement des expressions du principe plus général de MFE, et donc sous-jacentes à ce principe.

Introduction

Most river channels display a stable or relatively stable hydraulic geometry that the basic flow relationships of continuity, resistance and sediment transport cannot by themselves explain. Many cross sections with different geometries may satisfy particular flow and sediment transport requirements, and yet observations of natural channels show us that they tend to maintain characteristic shapes. As a result, extremal hypotheses have been proposed and tested, such as minimum entropy production (Leopold and Langbein 1962, Langbein 1964), minimum energy dissipation rate (e.g. Brebner and Wilson 1967, Yang and Song 1979, Yang et al 1981, Song and Yang 1982, Yang 1987), maximum sediment transporting capacity (MSTC) (e.g. Pickup 1976, Kirkby 1977, White et al 1981, 1982, Wang et al. 1986, Bettess and White 1987, Farias 1995), minimum stream power (MSP) (Chang 1979, 1980a,b, Millar and Quick 1993, 1998), maximum friction factor (Davies and Sutherland 1980, 1983), and minimum Froude number (Jia 1990, Yalin and Silva 1999, 2000). Each of these hypotheses is believed to represent a general principle within the fluvial system, the operation of which leads to the selection of a single preferred cross section out of many possibilities. In certain circumstances they are able to provide acceptable predictions of regime channel geometry, however, due principally to the lack of a convincing physical explanation for channel adjustment, their use has been a source of criticism (e.g. Griffiths 1984, Ferguson 1986, Darby and Thorne 1995, Knighton 1998).

Forms and patterns in natural systems are often the products of optimized circumstances and related inherently to operational efficiency (e.g. Stevens 1977, Hansen 1993, Casti 1998). In the systems of motion, optimum operational efficiency requires, by the minimization of a systems 'action', a least 'expenditure' in completing a particular task; out of many possible alternatives nature always follows a path that is the most 'economical'. This least action principle (LAP) was originally formulated by Maupertuis, Euler, Lagrange, Hamilton and Jacobi during 18th to 19th centuries and it has led to the establishment of the variational approach and analytical mechanics (Lanczos 1966, Konopinski 1969, Arnold 1989, Ginsberg 1995). In the 20th century, Richard Feynman identified the application of LAP in quantum physics and consequently established what is termed *fundamental* physics (Brown and Rigden 1993). The widespread applications of LAP outside mechanics and physics have also been documented (e.g. Zipf 1949, Dickel 1989, Amirikian 1992, Dunn and Laflamme 1993, Susperregi and Binney 1994), leading LAP to be conceived as a universal law (e.g. Bunn 1995, Casti 1998). The application of LAP in hydrodynamics has been investigated by Brenier (1989, 1993) and his promising findings have stimulated us to look at whether LAP is able to explain the formation of regular hydraulic geometry relations in fluvial systems.

This study shows that LAP is inherent in the self-adjusting behavior of alluvial channels. Channel flow is found to be capable of reaching a state of MFE (Maximum Flow Efficiency, defined here as the maximum sediment transporting capacity per unit available stream power), at which a stable channel section is constructed with the same geometry as those widely observed in na-

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ture. Importantly, this study offers support for the use of both maximum sediment transporting capacity (MSTC) and minimum stream power (MSP) concepts for they are subsumed by the more general principle of MFE, a direct expression of LAP in fluvial mechanics.

The least action principle and the shape adjustment of alluvial channel cross-sections

The least action principle (LAP) is an approach to characterize the *entire pattern* of a motion using scalar quantities (energy and work) without reference to *all the forces* acting on or within the system. By way of integral analysis, it examines the possible trajectories or paths of motions and demonstrates that motions occurring in nature always take the path of least action.

There are many expressions of LAP of which Hamilton's is the most general. Hamilton's LAP takes a form

$$I = \int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} L dt = \text{a minimum} \tag{1}$$

which requires the satisfaction of the following condition

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \tag{2}$$

where I represents the quantity of action, T is the kinetic energy, V is the potential energy, $L(=T\text{-}V)$ is the Lagrangian function, q_r is the path position variable, and $\dot{q}_r$ is the velocity of movement following the path. The time intervals t_1 and t_2 are fixed, as are also the coordinate sets at both starting and ending points of motion.

Given that no movement occurs in the direction of q_r, that is to say the system is independent of time and remains static, Eq. 2 becomes

$$\frac{\partial V}{\partial q_r} = 0 \tag{3}$$

Because potential energy V in Eqs. 1 to 3 decreases in the gravitationally downward direction, the typical position q_{rm} that satisfies Eq. 3 illustrates minimum values for both potential energy V and quantity of action I, or

$$\underset{q_r = q_{rm}}{I} = \text{a minimum} \propto \underset{q_r = q_{rm}}{V} = \text{a minimum} \tag{4}$$

If the system concerned is not subject to any external constraint, a stationary potential energy satisfies Eq. 4. When there are external constraints that are in the form of $\phi_r(q_r) = 0$, the minimum value of potential energy is normally greater than zero and depends on the constraints. As a special form of LAP, the principle of minimum potential energy (MPE) has been widely applied as a condition both necessary and sufficient for a conservative system (work done by the external forces is zero) to maintain a stable or dynamic equilibrium (e.g. Riley and Sturges 1993).

The energy of steady, uniform alluvial-channel flow is ultimately derived from gravitational potential energy, and alluvial channel flow is subject to a constraint of $\phi_r(q_r) = 0$, which consists of the known relationships of flow continuity, resistance and sediment transport. The specific expressions of the three relationships in alluvial channels apply to any form of channel section and normally include two position variables (channel width and depth) for given flow discharge, sediment load, channel slope and channel boundary composition. When the most basic channel form factor, width-depth ratio ζ, is for ease of illustration introduced, the following part of this study shows that the two position variables can be reduced into only one for a rectangular channel section. The velocity of movement $\dot{q}_r$ in the direction of q_r, i.e. ζ, will keep a constant of zero because channel flow moves uniformly only on a direction perpendicular to the variation of sectional shape factor ζ (Fig. 1). Since these conditions satisfy the principle of MPE, Eq. 4 for steady, uniform channel flow is written as

$$V(\zeta)\Big|_{\phi_r(\zeta)=0,\ \zeta=\zeta_m} = \text{a minimum} \tag{5}$$

where ζ_m defines the section that is in stable equilibrium or in regime (non-erodible and non-depositional).

Eq. 5 recognizes that if all the relevant cross-section positions are considered, the number of channel sections satisfying the physical relationships of flow continuity, resistance and sediment transport can actually be numerous. However only a single cross-section will satisfy Eq. 5 and also minimize potential energy and action. The solution to Eq. 5 is the dynamic equilibrium cross-section seen in alluvial channels. As alluvial-channel flow is maintained

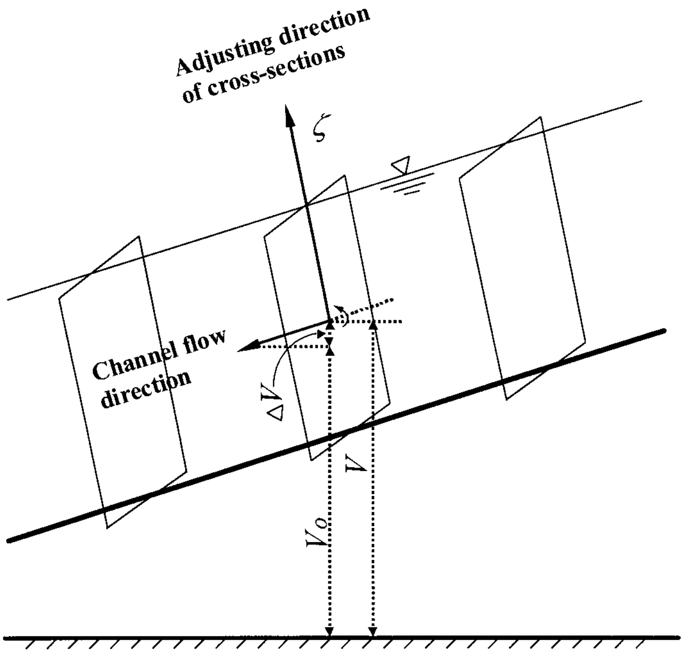

Fig. 1 Adjusting direction of channel cross-sections on a plane perpendicular to the channel flow direction

by an elevation difference ΔH, the potential energy is then equal to its varying part ΔV as

$$V = mg\Delta H = \gamma Q\Delta t \cdot \Delta H = \gamma Q\Delta t \cdot L \cdot S \tag{6}$$

If fluid mass m or flow discharge Q is given within the fixed time scale of Δt, the variation of ζ (width-depth ratio) occurring on a fixed length of channel L leads Eq. 5 to be easily determined by solving

$$\Omega(\zeta)\Big|_{\phi_r(\zeta)=0,\ \zeta=\zeta_m} = \text{a minimum} \tag{7}$$

where Ω is total stream power per unit channel length, or $\Omega = \gamma QS$. In most cases, flow discharge Q is an imposed factor and thus Eq. 7 becomes

$$S(\zeta)\Big|_{\phi_r(\zeta)=0,\ dQ=0,\ \zeta=\zeta_m} = \text{a minimum} \tag{8}$$

According to sediment transport theories, the rate of sediment transport in channels reflects the rate of work done by the available stream power Ω, and thus sediment discharge Q_s can be a surrogate of the energy available for sediment transport (e.g. Bagnold 1966). Therefore, the following relationship maintains

$$\frac{Q_s}{\Omega(=\gamma QS)} = \mathrm{F_e} \tag{9}$$

where $\mathrm{F_e}$ represents the efficiency of flow in transporting sediment load by available energy and is a function of channel width-depth ratio ζ, i.e. $\mathrm{F_e} = \mathrm{F_e}(\zeta)$. Therefore, it is inferred from Eq. 9 that in transporting a given amount of sediment Q_s, the only way to make Ω a minimum by changing ζ is to maximize $\mathrm{F_e}$, resulting in

$$\mathrm{F_e}(\zeta)\Big|_{\phi_r(\zeta)=0,\ dQ_s=0,\ \zeta=\zeta_m} = \text{a maximum} \tag{10}$$

On the other hand, Eq. 10 shows that the only way to keep Ω both minimum and constant over the whole range of variation in ζ is to maximize sediment transporting capacity Q_s in order to balance maximum flow efficiency, leading to

$$Q_s(\zeta)\Big|_{\phi_r(\zeta)=0,\ dQ=0,\ \zeta=\zeta_m} = \text{a minimum} \tag{11}$$

In these simple cases, logical deduction from the principle of least action clearly demonstrates the equivalence of minimum stream power (MSP) and maximum sediment transporting capacity (MSTC). Essentially MSP and MSTC are two sides of the same coin - the more general maximum flow efficiency (MFE). However, two questions arise; firstly, whether the above inferences are justified in terms of the basic flow relationships (i.e. whether the inferences produce theoretical results consistent with those shown by natural channels); and secondly, whether sediment transport theories different from Bagnold's stream power theory provide the same results.

Justification of theoretical inferences: a case study

In order to address these questions and to confirm that the theoretically inferred results are applicable to natural channels, the following case study presents a direct analysis of empirically derived relationships for resistance and sediment transport. The case study employs DuBoys' (1879) classic sediment transport formula that was developed from tractive-force theory. Alongside DuBoys' sediment transport formula our case study applies Lacey's flow resistance relationship, for two reasons. Firstly, Lacey's (1958) relationship was developed from studies of numerous stable canals and is applicable to channels that exhibit a mobile sand bed for sediment transport (Lacey 1958, Chitale 1966). Secondly, Chang (1980b) used these two relationships in his computer-based analytical approach incorporating his hypothesis of minimum stream power (MSP), and his results demonstrated the possibility of achieving stable channel geometry relations consistent with field observations.

Suppose A, D and R are the cross sectional area, mean depth and hydraulic radius of channel cross-sections, respectively, Lacey's (1958) flow resistance relationship is then written as

$$\frac{Q}{A} = \frac{1}{N_a} D^{1/4}\sqrt{RS} \tag{12}$$

where in SI units, coefficient N_a is related to sediment size d (*mm*) in a way of $N_a = 0.0253d^{1/8}$.

Suppose sediment transport occurs on a cross-section with W as bed width, DuBoys' (1879) sediment transport relationship is commonly written as

$$Q_s = C_d\tau_o(\tau_o - \tau_c)W \tag{13}$$

where $\tau_o(=\gamma RS)$ and τ_c are flow shear stress and critical shear stress for the incipient motion of sediment, respectively. In SI units, coefficient C_d and τ_c are determined by sediment size d(*mm*) in the form of $C_d = 0.17d^{-3/4}$ and $\tau_c = 0.061 + 0.093d$ (Chang 1980b).

Analysis is carried out assuming that channel flow is uniform and straight and the channel cross-section rectangular. For the transport of imposed water and sediment loads Q and Q_s with a given total stream energy Ω (=γQS) or on a constant of energy slope S, Eqs. 12 and 13 are therefore the functions of two cross-section geometric factors: width W and depth D. When a channel shape factor ζ, which is defined as the width/depth ratio, or $\zeta = W/D$, is introduced, Eqs. 12 and 13 are written as functions of ζ only. As a result, Eq. 12, in combination with cross-section geometric relations and the definition of $\tau_o = \gamma RS$, gives

$$W=\left(\frac{QN_a}{\sqrt{S}}\right)^{4/11}\zeta^{5/11}(\zeta+2)^{2/11};\quad \frac{dW}{d\zeta}=\frac{W}{11}\frac{7\zeta+10}{\zeta(\zeta+2)};$$
$$\tau_o=\gamma N_a^{4/11}S^{9/11}Q^{4/11}\frac{\zeta^{5/11}}{(\zeta+2)^{9/11}};\quad \frac{d\tau_o}{d\zeta}=\frac{4\tau_o}{11}\frac{2.5-\zeta}{\zeta(\zeta+2)} \tag{14}$$

Incorporating Eq. 14 into the differential form of DuBoys' sediment transport relationship of Eq. 13 yields

$$\frac{dQ_s}{d\zeta}=Q_s\left[\frac{1}{W}\frac{dW}{d\zeta}+\frac{2\tau_o-\tau_c}{(\tau_o-\tau_c)\tau_o}\frac{d\tau_o}{d\zeta}\right]$$
$$=\frac{Q_s}{11\zeta(\zeta+2)}\frac{(30-\zeta)\tau_o-(20+3\zeta)\tau_c}{\tau_o-\tau_c} \tag{15}$$

Let $dQ_s/d\zeta=0$ because Q_s is a given variable, and Eqs. 14 and 15 thereby produce the following solutions at a typical point $\zeta=\zeta_m$

$$\frac{\tau_o}{\tau_c}=\frac{20+3\zeta_m}{30-\zeta_m}\quad\text{or}\quad\frac{\tau_o-\tau_c}{\tau_c}=\frac{4(\zeta_m-2.5)}{30-\zeta_m};$$
$$Q_s=4K_1\tau_c\frac{\zeta_m^{10/11}}{(\zeta_m+2)^{7/11}}\frac{\zeta_m-2.5}{30-\zeta_m} \tag{16}$$

where coefficient K_1 is defined as

$$K_1=\gamma C_d N_a^{8/11}S^{7/11}Q^{8/11} \tag{17}$$

It is obtained in Eq. 15 that $dQ_s/d\zeta>0$ for $\zeta<\zeta_m$ but for $\zeta>\zeta_m$, $dQ_s/d\zeta>0$. At the point ζ_m, therefore, constant Q_s is actually a maximum that alluvial channels are capable of transporting

$$Q_s\big|_{\zeta=\zeta_m}=\text{a maximum} \tag{18}$$

Importantly, DuBoys' sediment transport formula in Eq. 13 under Lacey's flow resistance condition in Eq. 12 is found to be in the form of $F_3=Q_s/\Omega^{4/5}$ (Appendix I), which at the points ζ_m illustrates a generalized optimal flow condition as

$$F_e\left[=\frac{Q_s}{\Omega^{4/5}}\right]\Bigg|_{\zeta=\zeta_m}=\text{ a maximum} \tag{19}$$

It is noted in Eqs. 19 and 9 that the definitions of the flow efficiency factor F_e are different. This difference leads to define F_e generally as Q_s/Ω^{α} ($\Omega=\gamma QS$), in which α may vary with different combinations of flow resistance and sediment transport relations, 1.0 for Bagnold's (1966) sediment transport model and 4/5 for DuBoys (1879) sediment transport model under Lacey's flow resistance condition. This is not surprising because there have been numerous sediment transport and flow resistance models and none of them is based on the exactly same mechanisms. Taking this into account, Eq. 19, which is a finding from our direct analysis of basic flow relationships, can be seen as being consistent with those theoretically inferred results in Eqs. 7 to 11, and is evidence that the least action principle (LAP) and the principles of minimum potential energy (MPE) and maximum flow efficiency (MFE) are indeed a property of alluvial channel flow.

Since stream power Ω is the only available energy supplied to maintain steady, uniform channel flow, the variation of Ω against channel cross-sectional shape reflects the degree of resistance of the section for the transport of imposed sediment load Q_s. Therefore, the section at which Ω approaches a minimum is the one for which energy expenditure in overcoming channel-boundary resistance is the least, making the channel maximally efficient. This further validates the applicability of LAP to alluvial channel-form adjustments.

Most importantly, Eq. 19 is able to produce theoretical hydraulic geometry relations that are essentially consistent with field observations. Although the generalized hydraulic geometry relations are controlled by multiple variables, including flow discharge, channel slope, bank strength and channel average roughness, the effects of bank strength and channel average roughness are normally limited to certain ranges (Huang and Nanson 1995, 1997, 1998, Huang and Warner 1995, Julien and Wargadalam 1995, Huang 1996). As a result, hydraulic geometry relations can be illustrated predominantly as functions of flow discharge Q and energy slope S. In order to make a comparison between observed and theoretically derived hydraulic geometry relations, Eqs. 14 and 16 are combined so as to yield basic theoretical relations as

$$\frac{3\zeta_m+20}{30-\zeta_m}\frac{(\zeta_m+2)^{9/11}}{\zeta_m^{5/11}}\propto Q^{4/11}S^{9/11};$$
$$W\propto Q^{4/11}S^{-2/11}\zeta_m^{5/11}(\zeta_m+2)^{2/11} \tag{20}$$

The relations in Eq. 20 are complicated expressions but are real hydraulic geometry relations with ζ_m varying in the range of 2.5 to 30. This complexity is a consequence of the fact that the traditional power law relations of hydraulic geometry are simplified forms of the true relations (Richards 1982). To reflect how the theoretically derived hydraulic geometry relations adjust within the range, acceptably averaged relationships are obtained by assigning ζ_m with the integral numbers in the range of 3 to 29. As a consequence, the items including ζ_m in Eq. 20 are simplified as the power functions of ζ_m, and because all of those items are the monotonic functions of ζ_m, their approximate power functions have relatively high correlation coefficients as

$$\zeta_m^{1.915}\propto Q^{4/11}S^{9/11}\quad (r=0.9110)$$
$$W\propto Q^{4/11}S^{-2/11}\zeta_m^{0.605}\quad (r=1.0000) \tag{21}$$

which, together with relations of $W/D=\zeta$ and $A=WD$, determine the averaged theoretical hydraulic geometry relations that are shown in Table 1.

Table 1. Hydraulic geometry relations in relation to flow discharge and channel slope

Huang and coworkers' model*	Julien and Wargadalam's (1995) model**	This study
$W \infty Q^{0.501} S^{-0.156}$	$W \infty Q^{0.4\sim0.5} S^{-(0.2\sim0.25)}$	$W \infty Q^{0.478} S^{0.076}$
$D \infty Q^{0.299} S^{-0.206}$	$D \infty Q^{0.4\sim0.25} S^{-(0.2\sim0.125)}$	$D \infty Q^{0.289} S^{-0.350}$
$V \infty Q^{0.200} S^{0.362}$	$V \infty Q^{0.2\sim0.25} S^{0.4\sim0.375}$	$V \infty Q^{0.233} S^{0.274}$

* Huang and Warner (1995), Huang and Nanson (1995, 1998) and Huang (1996).

** Also see Huang (1996).

Table 1 also presents a comparison of the hydraulic geometry relations theoretically derived in this study with those developed based on a large set of field observations by Huang and coworkers (Huang and Warner 1995, Huang and Nanson 1995, 1998, Huang 1996) and by Julien and Wargadalam (1995) when the limited influences of other factors are ignored. Although the three sets of relations are obtained from totally different approaches, there are only minor differences between the exponents of flow discharge and channel slope. This consistency again suggests that natural alluvial channel-form adjusts in accordance with LAP, MPE and MFE.

Furthermore, the optimum flow condition defined in Eqs. 16 and 17 yields a complicated relationship among S, Q, Q_s and ζ_m, which can be averaged as

$$S \propto Q_s^{7/11} Q^{-8/7} \zeta_m^{-4.6500} \quad (r = 0.977) \tag{22}$$

Consequently, the combination of Eqs. 21 and 22 yields the optimum channel geometry relationships that are written into the functions of Q and Q_s as

$$\begin{aligned} W &\propto Q_s^{0.04} Q^{0.426} \propto \left(\frac{Q_s}{Q}\right)^{0.04} Q^{0.466}; \\ D &\propto Q_s^{-0.184} Q^{0.526} \propto \left(\frac{Q_s}{Q}\right)^{-0.184} Q^{0.342}; \\ S &\propto Q_s^{0.526} Q^{-0.678} \propto \left(\frac{Q_s}{Q}\right)^{0.526} Q^{-0.152} \end{aligned} \tag{23}$$

When the effects of Q_s/Q are ignored, the relations in Eq. 23 are very close to the traditionally cited regime relations of $W \infty Q^{1/2}$, $D \infty Q^{1/3}$ and $S \infty Q^{-1/7}$ (Lacey 1929, 1933, 1946, 1958, Ackers and Lacey 1992). This explains why in general terms the regime theory has been found applicable to canals and hydraulic geometry relations to natural streams because the later involves a wider range of variation in sediment concentration than the former.

Moreover, it is noted in Eq. 16 that it is the excess shear stress τ_o - τ_c that determines the range of variation in optimum channel shape ζ_m. When τ_o is at the threshold for the incipient motion of sediment, $\tau_o = \tau_c$ leading to ζ_m= 2.5. In contrast, when τ_o is very large such that τ_c appears so trivial as to be ignored without causing significant errors in the term τ_o - τ_c, ζ_m has a value of 30. This means that it is the bed load that plays a crucial role in constructing the stable channel geometries, and all of the theories on bedload transport can be expressed so as to include the term τ_o - τ_c as the determining factor.

Although our case study applies only specific flow resistance (Lacey's) and sediment transport (DuBoys') relations, the derived optimum channel geometry relations are consistent with those obtained from mathematical and computer-based analyses using a wide range of flow resistance and sediment transport (either bed load or bed material load) relations (e.g. Chang 1979, 1980b, White et al 1982, Wang et al. 1986, Farias 1995, Huang and Nanson 2000, 2001), and therefore these results are of general interest.

Discussion and conclusions

Though preliminary in nature, our research shows that the maximal efficiency of straight alluvial channels has its physical basis in the widely recognized physical principle of least action. This principle is general for flow in rivers and canals and causes alluvial channels in very different environments to exhibit remarkably regular hydraulic geometry relationships.

The application of the least action principle (LAP) to the adjustment of channel geometry is not artificial or teleological, but a soundly based and computationally economical method for the determination of alluvial channel geometry. Work in mechanics and physics shows clearly that the extremals of LAP are the exact solutions of Newton's equations of motion, and applying the principle is simply a different way of analyzing the same problems of motion. However, for complex systems that are short of a convincing force-based equation or are under complicated constraints, the application of LAP enables simpler and more economical analytical procedures for it does not need to account for *all the forces* acting on or within them (e.g. Lanczos 1966, Konopinski 1969, Arnold 1989, Brown and Rigden 1993, Ginsberg 1995). The analysis undertaken in this study, and the computer-based studies of Chang (1979, 1980a,b) and White et al. (1981), also show clearly that the application of LAP makes it possible to avoid difficulties in developing an additional purely empirical relation when seeking a closure solution for alluvial channel geometry.

The explicit use of LAP is an advance over the thermodynamic analogies and the empirical formulas previously used to justify extremal hypotheses, not only in terms of methodological rigor but also in that it clarifies the confusion regarding which of the numerous extremal hypotheses should be accepted. It is clear that the concept of maximal flow efficiency (MFE) embraces both maximum sediment transporting capacity (MSTC) and minimum stream power (MSP) and that results from an analysis based on any of MFE, MSTC and MSP would be highly consistent both with each other and with observations from natural fluvial systems. Secondly, it suggests that consideration of the extremal principles of MSTC, MSP and MFE should be included in canal design, for while a number of different sections may satisfy flow and sediment transport continuity, these channels will tend to adjust their forms towards the most efficient section. Indeed, Chang (1979, 1980a,b) and White et al (1981, 1982) in particular, have demonstrated that the use of both MSP and MSTC provides a practically convenient method for canal design.

As stated previously, that alluvial channels adjust to the maximally efficient section is demonstrated by the consistency between bankfull hydraulic geometry relationships of natural alluvial channels and those of the maximally efficient channels identified in this analysis (Table 1). Given the existence of large numbers of sediment transport and flow resistance formulas, it should not be surprising to find that in some cases the use of MFE, MSTC and MSP leads to results incompatible with field observations. However the assumptions used in analyses are critical in determining the results produced and so must be carefully evaluated. An example is given by the study of Griffiths (1984), in which five extremal hypotheses, including MSP and MSTC, were examined using the Lagrangian multiplier method and none was found to be capable of producing reasonable results. However, Griffiths assumed that the channel section is so wide that the hydraulic radius R is equal to average depth D. If this condition were generally applicable which it is not, the case study presented in this study would also yield the unreasonable result that MFE, MSTC and MSP occur when $\tau_o = -3\tau_c$.

Finally, the ideal scientific methodology is one that is both rational and empirical, in the sense that it is able to explain observed data through an understanding of physical causes rather than merely using correlations between variables. Such a methodology is likely to be the most fruitful in terms of providing explanation and prediction. To date, studies of hydraulic geometry have been largely inductive and focused on describing rather than on explaining the consistency seen in channel geometry. In outlining the applicability of LAP, we present a theory that is strongly supported from available geomorphic evidence, yet which can be further tested with ongoing research. It is logical to assume – at least as a starting point – that a single set of laws governs all alluvial channel forms. Given this, work that attempts to elucidate these laws deserves increased study, development and attention.

Appendix I. Derivation of flow efficiency factor F_e

To find the exact form of F_e in DuBoys' formula under Lacey flow resistance condition, Eq. 13 is transformed into

$$Q_s = c_d \tau_o (\tau_o - \tau_c) B = c_d \frac{\tau_o^{1+x}(\tau_o - \tau_c)}{\tau_o^x} B = \Omega^7 \frac{(\tau_o - \tau_c)}{\tau_o^x} \omega(\zeta) \tag{A1}$$

To determine the expressions of λ, x and ω(ζ) in Eq. A1, the B - ζ and τ_o - ζ relationships given in Eq. 14 under Lacey flow resistance condition are incorporated into Eq. A1, yielding:

$$7 = \frac{4}{5}; \quad x = \frac{1}{5}; \quad \omega(\zeta) = c_d \gamma^{2/5} n_a^{4/5} \frac{\zeta}{(\zeta+2)^{4/5}} \tag{A2}$$

As a result, DuBoys sediment transport in Eq. 13 under Lacey flow resistance condition in Eq. 12 can be written as:

$$\frac{Q_s}{\Omega^{4/5}} = F_e = \omega(\zeta) \frac{\tau_o - \tau_c}{\tau_o^{1/5}} \tag{A3}$$

Consequently, the variation of F_e against ζ can be illustrated by

$$\frac{dF_e}{d\zeta} = F_e \left[\frac{1}{\omega(\zeta)} \frac{d\omega(\zeta)}{d\zeta} + \left(\frac{1}{\tau_o - \tau_c} - \frac{1}{5\tau_o} \right) \frac{d\tau_o}{d\zeta} \right] = F_e \frac{(30-\zeta)\tau_o - (20+3\zeta)\tau_c}{11\zeta(\zeta+2)} \tag{A4}$$

It is noted in Eqs. 15 and A4 that $dQ_s/d\zeta/Q_s = dF_e/d\zeta/F_e$. Therefore, at the point $\zeta=\zeta_m$ where Q_s achieves a maximum, F_e, or the whole term of $Q_s/\Omega^{4/5}$, also reaches a maximum.

Notation

LAP least action principle
MFE maximum flow efficiency
MSTC maximum sediment transporting capacity
MSP minimum stream power
MPE minimum potential energy
I quantity of action
T kinetic energy
V potential energy
V_o potential energy corresponding to the base height of water body above datum
ΔV varying part of potental energy corresponding to channel flow depth
q_r path position variable
$\ddot{q}_r$ velocity of movement on the path
q_{rm} path position where least action and minimum potential energy are satisfied
$\phi_r(q_r)$ external constraints as functions of path position variable q_r
ζ width/depth ratio as a position variable
ζ_m width/depth ratio for a section in dynamic equilibrium or in the most efficient state
m mass of fluid
g acceleration due to gravity
γ fluid specific weight
Q flow discharge
S energy gradient or channel slope

Δt time scale difference
ΔL channel length difference
ΔH elevation difference
Q_s sediment discharge on total channel width
F_e flow efficiency in transporting sediment ($=Q_s/\Omega^{\alpha}$)
Ω total stream power ($=\gamma QS$)
α exponent
A cross-sectional area
W channel width
D mean channel depth
R hydraulic radius
d sediment size
N_a coefficient in Lacey's flow resistance relation
Cd coefficient in DuBoys' bedload transport formula
τ_o flow shear stress
τ_c critical flow shear stress for the incipient motion of sediment
K_1 coefficient
r correlation coefficient

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