Modelling of supercritical flow conditions revisited; NewC Scheme

Une nouvelle perspective pour la modélisation des écoulements torrentiels, le schéma NewC

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SUMMARY
A hydrodynamic numerical model for one-dimensional free-surface flows, named 'NewC', is presented. NewC is a finite difference scheme which has a major advantage over schemes used currently in engineering applications in that, while the algorithmic structure is of the subcritical-flow-type, it is capable of modelling subcritical, supercritical and transcritical flow conditions without requiring any changes to the governing equations. The scheme is shown to be unconditionally stable for a range of Courant numbers even for Froude numbers greater than or equal to one. The computational effort expended compares favourably with the finite difference schemes used currently. The NewC scheme has the additional advantage that it is relatively straightforward to incorporate into algorithms for the solution of flows in free-surface networks.

RÉSUMÉ
On présente ici un modèle numérique hydrodynamique, nommé NewC, pour les écoulements unidimensionnels à surface libre. NewC est un schéma aux différences finies qui présente l'avantage majeur, sur les schémas utilisés habituellement dans les applications industrielles, d'être capable de traiter les écoulements fluviaux, torrentiels et critiques sans changer quoi que ce soit dans les équations, alors que la structure algorithmique est de type écoulement fluvial. On montre que le schéma est inconditionnellement stable pour une gamme de nombres de Courant et même de nombres de Froude supérieurs ou égaux à un. Le coût calcul se compare favorablement avec ceux des schémas aux différences finies utilisés couramment. Le schéma NewC a de plus l'avantage de pouvoir être facilement introduit dans les algorithmes de résolution des réseaux d'écoulements à surface libre.

Introduction

Difficulties that arise in solving for supercritical flow conditions using the method of finite differences date back to late sixties when implicit numerical schemes were first used in the solution of one-dimensional free-surface flows. In contrast to explicit schemes, implicit ones require a defined algorithmic structure (Abbott and Minns, 1997, p.245). Upstream control of supercritical flows is reflected in a requirement for two boundary conditions at the upstream end, which results in one type of algorithmic structure (single sweep). Subcritical flows, on the other hand, require one boundary condition at each end of the domain and, hence, a different algorithmic structure (double sweep). When dealing with separate supercritical or subcritical flow situations satisfactory models can be built using an appropriate algorithmic structure but such models are not very practical as they cannot deal with so-called transcritical flows; flows that change their character within the considered domain.

Satisfactory solutions for the transcritical flow case can be obtained using the method of finite volumes (Weiyan, 1992, pp. 241-242; Zhao et al., 1994; Hu et al., 1998). Its integral formulation makes it possible to obtain the weak solution which enables the modelling of discontinuities, while solution of the Riemann problem at cell boundaries is used to deal with solving for different flow conditions. However, this method, although widely used by the research community, has still not been adopted for commercial engineering software. Instead, all the major commercially available software packages for solution of one-dimensional free-surface flows (i.e. river systems, sewer networks) are based on the method of finite differences, more precisely on implicit finite difference schemes. The reason for this choice is that models built upon implicit schemes have proven to be numerically efficient, stable and robust. As such they are suitable for use by practising engineers who are not specialists in Computational Hydraulics. Due to the difficulties in solving transcritical flows mentioned above these commercial packages either cannot deal with supercritical flows or solve them in some approximate way (Kutija, 1993).

In recent years some authors have claimed to have developed implicit finite difference models which are able to deal with supercritical and transcritical flows (Stelling, private communication; Schuurmans and Nelan, 1996; Zhong Ji, 1998). Unfortunately, in these papers, details of the schemes used are either not given or there is no justification offered for these claims.

In this paper an implicit numerical scheme, named NewC, is described and analysed to show how it maintains the same level of numerical efficiency and unconditional stability as the schemes currently used in the commercial packages while enabling solution of supercritical and transcritical flows.

Analysis of the problem and some earlier solutions

One dimensional free surface flows are described by the de Saint Venant equations:

$$\text{Continuity:}\quad b_s \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (1)$$

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Momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\beta \frac{Q^2}{A}\right) + gA\frac{\partial h}{\partial x} + gA\frac{Q\,|Q|}{K^2} = 0 \qquad (2)$$

where:
Q – discharge (m^3/s)
h - water level* (m above sea level)
b_s - storage width (m)
A - cross-sectional area (m^2)
β - Boussinesq coefficient
g - gravity acceleration (m^2/s)
K – conveyance (m^3/s)

Finite difference schemes used principally for the solution of these equations utilise either staggered or non-staggered discretisation grids. Non-staggered grids have the two dependent variables placed at the same discretisation points while staggered grids alternate the dependent variables (see Fig.1). The most widely used numerical schemes are the Preissmann scheme on non-staggered grids (Preissmann, 1960; Liggett and Cunge, 1975) and Abbott-Ionescu scheme on staggered grids (Abbott and Ionescu, 1967; Liggett and Cunge, 1975). Both of these schemes offer a linearised discretisation of the governing equations resulting in a system of linear simultaneous equations to be solved for each time step. In the case of the Abbott-Ionescu scheme the system matrix is tridiagonal while in the case of Preissmann scheme the system matrix is pentadiagonal. When accompanied by proper boundary conditions these systems of equations are usually solved by a direct, efficient method called the double sweep. It is a form of Gaussian elimination for banded matrices. The usual requirement for the system matrix to be solvable by this method is that it is diagonally dominant (Volkov, 1986, pp. 155-160).
The system of equations obtained by the finite difference approximation has two unknowns more than the number of equations. Adding two equations representing the boundary conditions solves this problem. Where these two equations are incorporated into the original system depends on the flow conditions. For subcritical flow one equation is added at each end, while for supercritical flow conditions they are both added at the upstream end. The consequence is that, in these two cases a different diagonal becomes the main diagonal of the system.

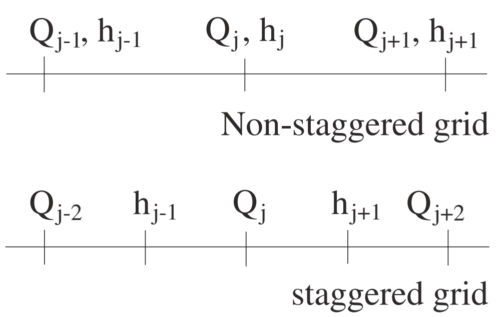


Fig. 1 Finite difference grids

*Note that h is the water level throughout this paper and that y is used for water depth

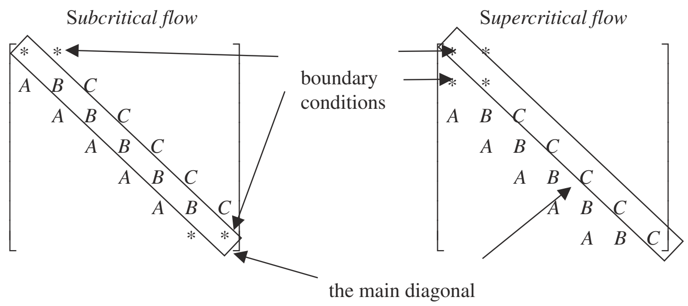
Let us consider the Abbott-Ionescu scheme. Upon finite difference approximation the general form of the equations (continuity and momentum) is:

$$A_j\ Y_{j-1}^{n+1} + B_j\ Y_j^{n+1} + C_j\ Y_{j+1}^{n+1} = D_j \qquad (3)$$

where Y_j^{n+1} represents unknown dependent variables (Q and h alternately) at the new time level $n+1$, while coefficients A, B, C and D depend on the flow conditions at the previous time step, discretisation steps in both directions (space and time) and other parameters. Index j denotes the position of the discretisation point around which this equation was approximated.
For an example with seven discretisation points the system of equations without the boundary conditions is:

$$\begin{bmatrix} A_1 & B_1 & C_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & C_2 & 0 & 0 & 0 \\ 0 & 0 & A_3 & B_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & A_4 & B_4 & C_4 & 0 \\ 0 & 0 & 0 & 0 & A_5 & B_5 & C_5 \end{bmatrix} \begin{bmatrix} Y_0^{n+1} \\ Y_1^{n+1} \\ Y_2^{n+1} \\ Y_3^{n+1} \\ Y_4^{n+1} \\ Y_5^{n+1} \\ Y_6^{n+1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix}$$

Now, two cases are considered; one with subcritical flow and another with supercritical flow and their system matrices are presented:



Comparing these two situations one can see that in the case of subcritical flow the main diagonal consists mainly of coefficients B while in the supercritical case it consists of coefficients C. Expressions for all the coefficients (A, B, C and D) depend upon the flow conditions, so their magnitude is flow related. In subcritical flow conditions coefficient B is predominant while in supercritical flow conditions coefficient C takes that role. (This is just a rough explanation. The situation is not so simple as alternate

rows originate from different equations*
From this example it should become obvious what happens if one tries to solve for supercritical flow conditions using the algorithmic structure appropriate for subcritical flows with one boundary condition each side. The main diagonal in that case consists of B coefficients while the C coefficients are the dominant ones so the matrix of the system is not diagonally dominant. Hence, there is no solution.
An obvious way to solve the problem of transcritical flows is to trace the point at which the flow conditions change and impose internal conditions that take care of providing the correct algorithmic structure for different flow types. Models based on this idea have been developed but have proven to be computationally too demanding to be used in engineering tools. Therefore, efforts were mainly concentrated on development of so-called through methods (Abbott, Havnø and Lindberg, 1991; Savic and Holly, 1993; Abbott and Minns, 1997; pp.243-248). The background to these methods is the use of the integral form of the conservation laws that enables finding weak solutions. The finite volume method is based on this idea and, hence, provides solutions for discontinuous flows. However, the finite difference method uses the differential form of the conservation laws and does not provide weak solutions (Abbott and Minns, 1997; pp.370-374).
It has been shown that partial or total reduction of the convective momentum term from the momentum equation changes the characteristic structure of the flow (Kutija, 1993; Fread, 1993). This enables solution for supercritical flow conditions with a subcritical-flow-like algorithmic structure (Abbott and Minns, 1997; p. 245). In terms of the system of equations this means that coefficients of the discretised equation (3) change in the supercritical flow case. However, when a reduced convective momentum term is implemented, the coefficients B_j remain dominant even for supercritical flow conditions. Hence, the matrix of the system with one boundary condition at each end stays diagonally dominant for all flow conditions i.e. condition (4) is fulfilled.
Although it has been included in some commercially very successful software packages, this method has its problems. The results for supercritical flow conditions overestimated water levels and delayed peaks (Kutija, 1993). The other down side is that additional calculations must be performed in order to obtain the Froude number for every discretisation point at every time step as the reduction coefficient for the convective momentum term has to be calculated on the basis of the local Froude number.

*Kutija (1993) has shown that the total diagonal dominance represents too severe condition that is not fulfilled even in the case of subcritical flow. The alternating equations originate from the momentum and continuity equations and, while one of them shows clear diagonal dominance the other does not. Therefore, an alternative method was devised combining any two consecutive equations and following the elimination procedure. For details see Kutija 1993. The necessary condition for a system of equations of the type (3) with one boundary condition at each end of the domain is

$$|C_j| - |A_j| \le 0 \tag{4}$$

for every equation in the system.

NewC scheme

The NewC numerical scheme is based on a staggered grid and bears some resemblance to the Abbott-Ionescu scheme. It uses the same algorithm (the double sweep algorithm) for both supercritical and subcritical flow conditions but does not require any alteration of the governing equations.
The discretisation grid is staggered with Q points placed at the beginning and the end of the domain* as shown in Fig.2.

Continuity Equation

$$b_s \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

The approximation is made according to the scheme sketched in Fig.3 with a weighting θ in the time dimension giving the difference equation:

$$h^{n+1}_{j+½} - h^{n}_{j+½} + \frac{\theta \Delta t}{\left(b_s\right)^{n+½}_{j+½} \Delta x}\left(Q^{n+1}_{j+1} - Q^{n+1}_{j}\right) + \frac{(1-\theta)\Delta t}{\left(b_s\right)^{n+½}_{j+½} \Delta x}\left(Q^{n}_{j+1} - Q^{n}_{j}\right) = 0$$

The only difference between this and the standard Abbott-Ionescu scheme is the form of the difference equation. We have:

$$h^{n+1}_{j+½} = \alpha^{n+1}_{j+½} Q^{n+1}_{j} + \delta^{n+1}_{j+½} Q^{n+1}_{j+1} + \gamma^{n+1}_{j+½} \tag{5}$$

where the coefficients are:

$$\alpha^{n+1}_{j+½} = \frac{\Delta t \theta}{\Delta x \left(b_s\right)^{n+½}_{j+½}}$$

$$\delta^{n+1}_{j+½} = -\frac{\Delta t \theta}{\Delta x \left(b_s\right)^{n+½}_{j+½}}$$

$$\gamma^{n+1}_{j+½} = h^{n}_{j+½} - \frac{\Delta t (1-\theta)}{\Delta x \left(b_s\right)^{n+½}_{j+½}} \left(Q^{n}_{j+1} - Q^{n}_{j}\right)$$

Momentum Equation

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta \; Q^2}{A}\right) + gA\frac{\partial h}{\partial x} + gA\frac{Q|Q|}{K^2} = 0$$

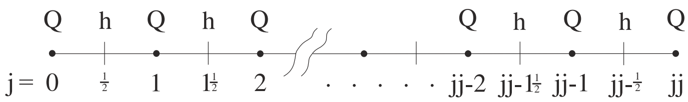


Fig. 2 Discretisation grid for NewC scheme

*When a network solution is sought the vertices of the network contain one h point in the centre and as many Q points as there are incident edges. These Q points coincide with the first or last Q point for each edge. At the vertex points continuity equations are placed. For more details about network solutions see Kutija (1995).

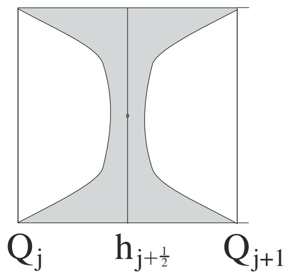


Fig. 3 Numerical operator for continuity equation

The approximation of the momentum equation is the major difference between the NewC scheme and the Abbott-Ionescu scheme. As the Abbott-Ionescu scheme spans two h points and just one Q point it cannot approximate the two partial derivatives with the respect to space (second and third term in the above equation). Therefore, the Abbott-Ionescu scheme uses the other form of the conservation equations, the algorithmic form (Liggett and Cunge, 1975, Abbott and Minns, 1997; Kutija, 1996) where each equation has a maximum of two derivatives with respect to time and only one with respect to space.

The NewC scheme spans three Q-points and two h-points as shown in Fig. 4 and it can use the Eulerian form of the governing equations. Upon finite difference approximation, the finite difference equations for each Q-point are given by:

$$Q_j^{n+1} - Q_j^n + \frac{\beta\ \Delta t\ Q_j^{n+\frac{1}{2}}}{\Delta x\ A_j^{n+\frac{1}{2}}}\left(\begin{array}{l}\theta\left(Q_{j+1}^{n+1}-Q_{j-1}^{n+1}\right)+\\ \left(1-\theta\right)\left(Q_{j+1}^{n}-Q_{j-1}^{n}\right)\end{array}\right)+$$
$$gA_j^{n+\frac{1}{2}}\frac{\Delta t}{\Delta x}\left\{\theta\left(h_{j+\frac{1}{2}}^{n+1}-h_{j-\frac{1}{2}}^{n+1}\right)+\left(1-\theta\right)\left(h_{j+\frac{1}{2}}^{n}-h_{j-\frac{1}{2}}^{n}\right)\right\}+$$
$$\frac{g\ \ A_j^{n+\frac{1}{2}}\Delta t}{\left(K_j^{n+\frac{1}{2}}\right)^2}\left|Q_j^n\right|Q_j^{n+1}=0$$

This can be written in the form:

$$\chi_j^{n+1}Q_{j-1}^{n+1}+\varepsilon_j^{n+1}h_{j-\frac{1}{2}}^{n+1}+\phi_j^{n+1}Q_j^{n+1}+\eta_j^{n+1}h_{j+\frac{1}{2}}^{n+1}+\lambda_j^{n+1}Q_{j+1}^{n+1}=\mu_j^{n+1} \qquad (6)$$

However, expression (5) can be used to eliminate the second and fourth term in equation (6), which results in the following, more compact, form:

$$a_j^{n+1}Q_{j-1}^{n+1}+b_j^{n+1}Q_j^{n+1}+c_j^{n+1}Q_{j+1}^{n+1}=d_j^{n+1} \qquad (7)$$

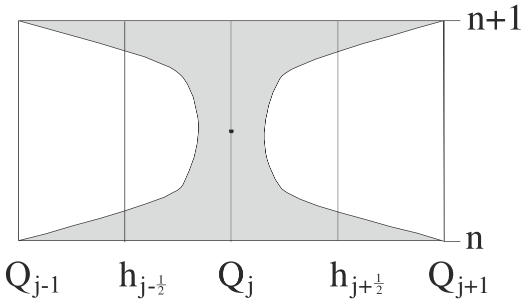


Fig. 4 Numerical operator for momentum equation

where the coefficients are:

$$a_j^{n+1}=-\frac{\beta\Delta t\theta}{\Delta x}\frac{Q_j^{n+\frac{1}{2}}}{A_j^{n+\frac{1}{2}}}-\frac{\Delta t\ \ \theta}{\Delta x}gA_j^{n+\frac{1}{2}}\alpha_{j-\frac{1}{2}}^{n+1}$$

$$b_j^{n+1}=1+\Delta t g\frac{A_j^{n+\frac{1}{2}}}{\left(K_j^{n+\frac{1}{2}}\right)^2}\left|Q_j^n\right|+\frac{\Delta t\theta}{\Delta x}gA_j^{n+\frac{1}{2}}\left(\alpha_{j+\frac{1}{2}}^{n+1}-\delta_{j-\frac{1}{2}}^{n+1}\right)$$

$$c_j^{n+1}=\frac{\beta\Delta t\theta}{\Delta x}\frac{Q_j^{n+\frac{1}{2}}}{A_j^{n+\frac{1}{2}}}+\frac{\Delta t\theta}{\Delta x}gA_j^{n+\frac{1}{2}}\delta_{j+\frac{1}{2}}^{n+1}$$

$$d_j^{n+1}=Q_j^n-gA_j^{n+\frac{1}{2}}\frac{\Delta t}{\Delta x}\left[\theta\left(\gamma_{j+\frac{1}{2}}^{n+1}-\gamma_{j-\frac{1}{2}}^{n+1}\right)+\left(1-\theta\right)\left(h_{j+\frac{1}{2}}^{n}-h_{j-\frac{1}{2}}^{n}\right)\right]$$
$$-\frac{\beta\Delta t}{\Delta x}\frac{Q_j^{n+\frac{1}{2}}}{A_j^{n+\frac{1}{2}}}\left(1-\theta\right)\left(Q_{j+1}^{n}-Q_{j-1}^{n}\right)$$

Note that the cross-sectional area, A, and the conveyance, K can only be calculated where h is known (the lines with spacial subscript of the form $j \pm ½$) thus average values must be used elsewhere, i.e.

$$A_j^n=½\ \left\{A_{j-\frac{1}{2}}^n+A_{j+\frac{1}{2}}^n\right\}\quad K_j^n=½\ \left\{K_{j-\frac{1}{2}}^n+K_{j+\frac{1}{2}}^n\right\}$$

Solution Method

A part of the solution method was already covered in the approximation of the momentum equation. In order to make it clear it will be explained in more detail here. If equations (5) and (6) are used the system of equations has the following form:

$$\begin{bmatrix}
* & * & & & & & & & & & \\
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\end{bmatrix}
\begin{bmatrix} Q_0^{n+1} \\ h_{\frac{1}{2}}^{n+1} \\ Q_1^{n+1} \\ h_{1+\frac{1}{2}}^{n+1} \\ Q_2^{n+1} \\ h_{2+\frac{1}{2}}^{n+1} \\ Q_3^{n+1} \\ h_{3+\frac{1}{2}}^{n+1} \\ Q_4^{n+1} \\ h_{4+\frac{1}{2}}^{n+1} \\ Q_5^{n+1} \end{bmatrix}
=
\begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} \qquad (8)$$

From the form of the matrix of the system we can see that the elimination of the even equations does not bring any fill-ins. It effectively reduces the system of equations to a much smaller system of equations. The form of this reduced system of equations is presented in (9). It almost halves the number of equations, with half as many unknowns as the original system of equations. Its unknowns are only discharges and its equations are only the ones originating from the momentum equations. This new system of equations consists of equations of type (7) as the expressions of type (5) were already incorporated in the coefficients of the equation (7) by substituting unknown water levels.

$$\begin{bmatrix} * & * & & & & \\ * & * & * & & & \\ & * & * & * & & \\ & & * & * & * & \\ & & & * & * & * \\ & & & & * & * \end{bmatrix} \begin{bmatrix} Q_0^{n+1} \\ Q_1^{n+1} \\ Q_2^{n+1} \\ Q_3^{n+1} \\ Q_4^{n+1} \\ Q_5^{n+1} \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix} \quad (9)$$

The reduced system of equations (9) is tri-diagonal and its solution is obtained by the double sweep algorithm (Abbott and Minns, 1997, pp. 252-255). This solution provides us with the discharges and an additional substitution sweep through all the equations (5) is performed to give the solution for the water levels. It should be noted that the NewC scheme does not require any additional computational effort compared with the Abbott-Ionescu scheme. Moreover, the double sweep algorithm is performed on half the number of equations but the additional substitution sweep and somewhat longer expressions for the coefficients can be seen as cancelling that advantage.

Stability analysis

Stability of the NewC scheme

Linearised stability analysis using Fourier series expansions was applied to the NewC scheme. Each variable in the finite difference equations (5) and (7) is replaced by its Fourier series expansion; we write;

$$h_j^n = \sum_{k=1}^{N} H_k^n e^{ik\alpha j} \quad \text{and} \quad Q_j^n = \sum_{k=1}^{N} \xi_k^n e^{ik\alpha j}$$

where H_k^n, ξ_k^n are the Fourier coefficients for h and Q respectively for wave number k at time level n, $\alpha = \pi\Delta x/l$ is the dimensionless wave number.

Only the k^{th} component of the expansion is then considered. Substitution into equations (5) and (7) results in a matrix equation. we have

$$\begin{bmatrix} H_k \\ \xi_k \end{bmatrix}^{n+1} = \mathbf{A} \begin{bmatrix} H_k \\ \xi_k \end{bmatrix}^{n}$$

where **A** is described as the amplification matrix for the scheme. Finally, the eigenvalues of **A**, λ, are determined and the requirement that $|\lambda| \le 1$ is imposed to obtain the stability criterion.

Stability analysis shows that the scheme is unconditionally stable for all the combinations of Courant and Froude numbers provided that θ is between 0.5 and 1. For $\theta = 0.5$ all of the eigenvalues have modulus 1, indicating that there is no amplification error.

The phase error is represented by the relative celerity (RC) defined as the ratio between the numerical and continuum celerity;

$$RC = \frac{\arctan\{\mathrm{Im}(\lambda)/\mathrm{Re}(\lambda)\}}{\pi Cr/NX}$$

where Cr is the Courant number defined as

$Cr = \left(u + \sqrt{gh}\right)\dfrac{\Delta t}{2\Delta x} = \left(\dfrac{Q}{A} + \sqrt{\dfrac{gA}{b}}\right)\dfrac{\Delta t}{2\Delta x}$ and Fr is the Froude number defined as $Fr = u/\sqrt{gh}$.

The phase portrait obtained for the case $\theta = 0.5$ is shown in Fig. 5. It can be seen that different values of Cr and Fr significantly influence the relative celerity. Phase and amplification portraits for the case $\theta = 0.7$ are given in Figs. 6 and 7 since this weighting is widely used in the examples presented below.

Stability of the algorithmic structure

The NewC scheme uses the subcritical algorithmic structure (one boundary condition at each end) for all flow conditions. Hence, the coefficient B will always be on the main diagonal. In order to check the stability of the solution method the criteria (4) devised by the first author and mentioned earlier was applied:

$$\left|c_j\right| - \left|a_j\right| \le 0 \quad \text{or} \quad \left|c_j\right| \le \left|a_j\right|$$

The coefficients c_j^{n+1} and a_j^{n+1} with substituted expressions for α_j^{n+1} and β_j^{n+1} are:

$$a_j^{n+1} = -\frac{\beta\Delta t\theta}{\Delta x}\frac{Q_j^{n+½}}{A_j^{n+½}} - \frac{\Delta t^2\theta^2}{\Delta x^2}\frac{gA_j^{n+½}}{b_{j-½}^{n+½}},$$

$$c_j^{n+1} = \frac{\beta\Delta t\theta}{\Delta x}\frac{Q_j^{n+½}}{A_j^{n+½}} - \frac{\Delta t^2\theta^2}{\Delta x^2}\frac{gA_j^{n+½}}{b_{j+½}^{n+½}}$$

They can be approximated as follows:

$$a_j^{n+1} \approx -2\frac{\beta\,\theta\,Fr\,Cr}{Fr+1} - 4\theta^2\frac{Cr^2}{(Fr+1)^2} \quad \text{and}$$

$$c_j^{n+1} \approx 2\frac{\beta\,\theta\,Fr\,Cr}{Fr+1} - 4\theta^2\frac{Cr^2}{(Fr+1)^2}$$

From the approximate expressions for the coefficients a_j^{n+1} and c_j^{n+1}, one can see that condition (4) is always fulfilled irrespective

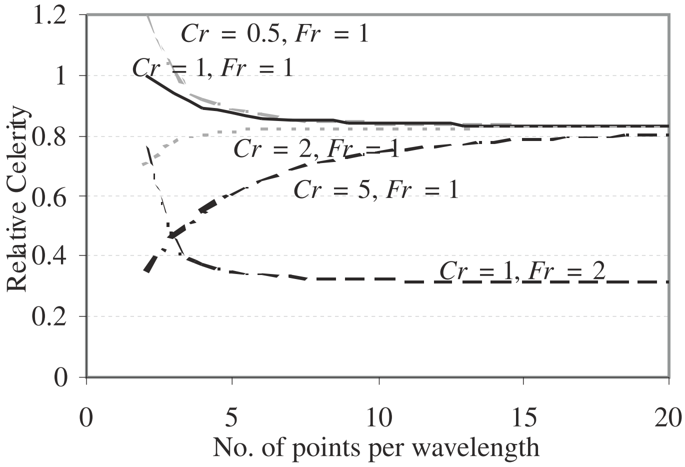

Fig. 5 Phase portrait for $\theta = 0.5$

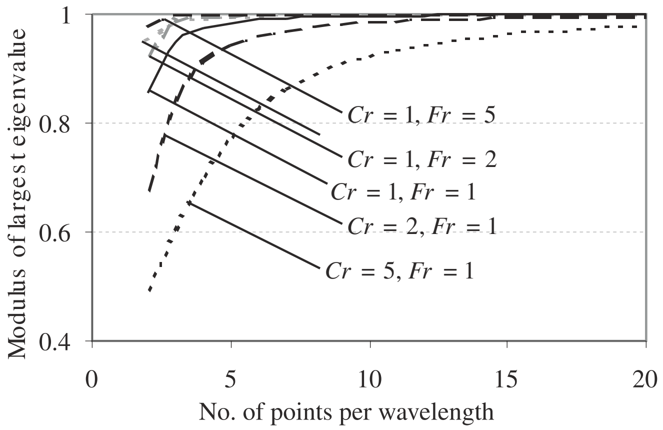

Fig. 6 Amplification portrait for θ = 0.7

of the flow conditions. Hence, the double sweep algorithmic structure with one boundary condition at each end of the domain is stable for subcritical, critical and supercritical flows.

Examples

Three groups of tests performed with the NewC scheme are presented. In the first group of tests (EXAMPLE 1) uniform flow in channels with mild and steep bottom slopes was tested for a variety of discharges. In the second group of tests (EXAMPLE 2) the influence of different downstream boundary conditions on supercritical flow is shown while in the third group (EXAMPLE 3) cases of transcritical flow are presented. All of the results presented are steady state conditions achieved under constant boundary conditions starting from different initial conditions. Details of the initial and boundary conditions are given in the results tables below. In all of the examples channel length is 1000m, the cross-section is rectangular with width 100m and the Chezy coefficient is 50 $m^{0.5}/s$.

The above table shows results of six tests involving different discharges and bottom slopes. All of the tests arrived at uniform flow conditions within less than 500 seconds and stayed stable for a long simulation time.

It can be seen in Fig. 8 that the influence of the downstream boundary conditions does not spread far upstream (in the example presented just for one Δx).

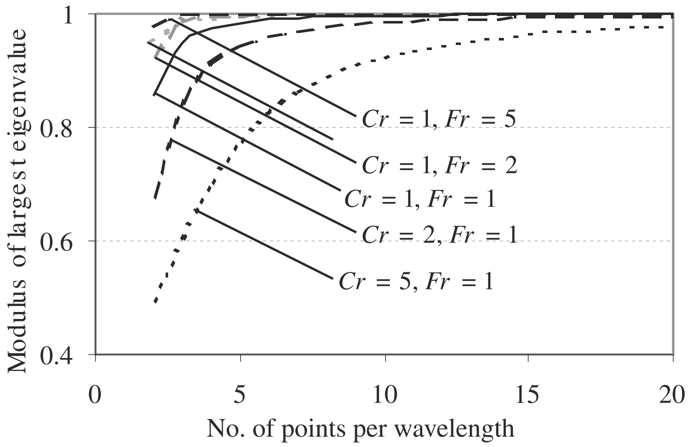

Fig. 7 Phase portrait for θ = 0.7

EXAMPLE 1 – uniform flow

DATA:	RESULTS					
	Q (m^3/s)	**h_{cr}** (m)	**critical slope**	**slope**	**h** (m)	**Fr**
channel geometry: bottom slope = uniform						
numerical parameters:	50	0.29	.0144	.002	0.37	0.71
Δx = 20m, Δt = 5,10,20,40 s; θ = 0.7, Cr = 1 - 16	100	0.47	.0088	.01	0.34	1.59
initial conditions:	250	0.86	.0048	.02	0.50	2.25
Q_i = constant throughout; y = 0.5m or 1m throughout	500	1.36	.0032	.03	0.70	2.75
boundary conditions: upstream – constant Q	1000	2.1	.0023	.04	1.00	3.16
downstream – rating curve Q=5000 $slope^{1/2}$ $y^{3/2}$	1500	2.84	.0016	.05	1.13	3.99

EXAMPLE 2 – different downstream boundary conditions

DATA:

channel geometry:
bottom slope = 0.02
numerical parameters:
Δx = 20m, Δt = 10s
θ = 0.7, Cr = 2.26
initial conditions:
Q_i = 500 m^3/s throughout
y = 0.6 m throughout
boundary conditions:
upstream – Q = 500 m^3/s
downstream
a) rating curve Q = 500 $y^{3/2}$
b) h = 0.6m
c) h = 1.0 m

Fig.9 and Fig.10 show transcritical flow profiles that occur due to the variation of bottom slope between mild and steep ones. As can be seen in both figures transitions between subcritical and supercritical flows and vice versa are smooth.

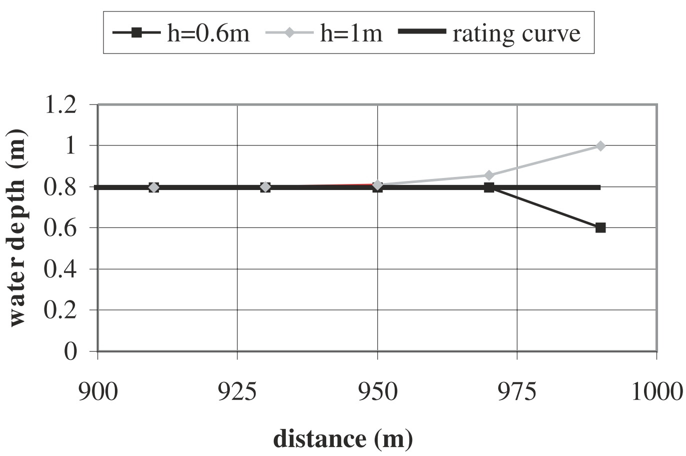

Fig. 8 Flow profile for EXAMPLE 2

EXAMPLE 3 – transcritical flow conditions

DATA:

a)
channel geometry:
300m with slope of 0.0033
400m with slope of 0.0510
300m with slope of 0.0016
numerical parameters:
Δx = 20m, Δt = 10s
θ = 0.7, Cr = 2
initial conditions:
Q_i = 1500 m^3/s throughout
y = 1.5 m throughout
boundary conditions:
upstream – Q = 1500 m^3/s
downstream
rating curve Q = 200 $y^{3/2}$

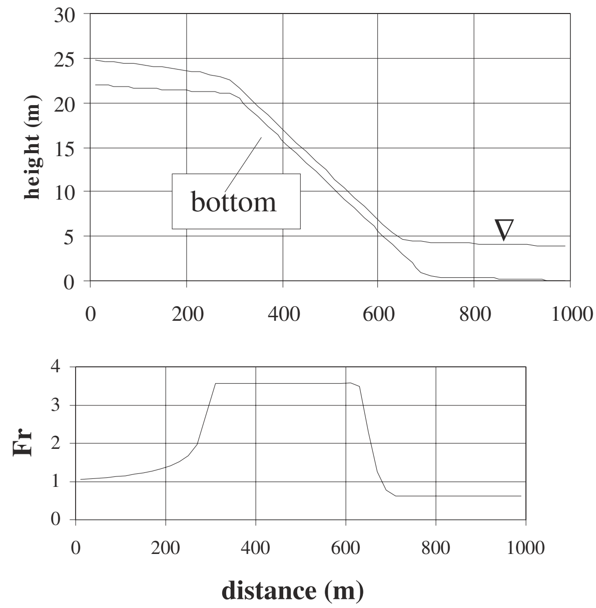

Fig. 9 Flow profile and Fr for EXAMPLE 3a

DATA:

b)
channel geometry:
300m with slope of 0.025
400m with slope of 0.0012
300m with slope of 0.033
numerical parameters:
Δx = 20m, Δt = 10s
θ = 0.7, Cr = 2
initial conditions:
Q_i = 1500 m^3/s throughout
y = 1.5 m throughout
boundary conditions:
upstream – Q = 1500 m^3/s
downstream
rating curve Q = 908.3 $y^{3/2}$

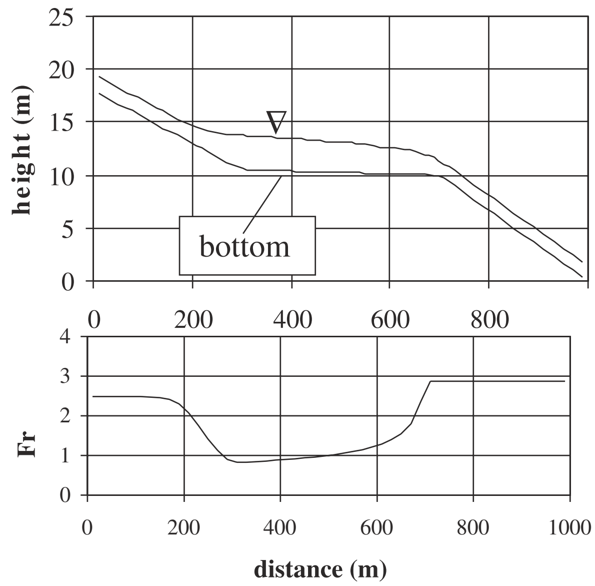

Fig. 10 Flow profile and Fr for EXAMPLE 3b

Discussion

It is interesting to note that the NewC scheme has something in common with the method of finite volumes on a staggered grid (Versteeg and Malalasekera, 1995, pp.139-142 and pp.169-171). The two schemes use the same five variables in discretising the momentum equation; NewC uses discharges at points *j-1*, *j* and *j+1* and water levels at points *j-½* and *j+½* while the finite volume method uses the discharge at point P, the water depths at the two cell faces (*w* and *e*) and the two discharges at the neighbouring points *W* and *E*. See Fig. 11.

It is thought that the ability of the NewC scheme to deal with transcritical flows follows in part from this parallel with the finite volume method. However, there are two important differences between the two formulations; First, the NewC scheme solves the differential form of the conservation laws while the finite volume method uses an integral formulation; second, the convective momentum term is treated differently by the two methods.

The NewC scheme has some limitations in solving for supercritical flow because of the way the boundary conditions are dealt with. The NewC scheme requires one boundary condition at each end of the domain. Hence, supercritical flow on a flat bot-

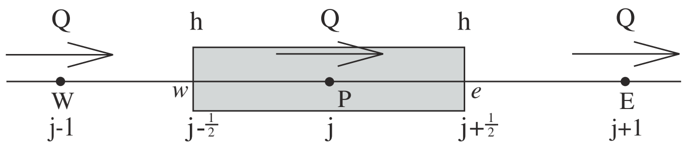

Fig. 11 Comparison of discretisation schemes for NewC and the finite volume method

tom, such as that found downstream of a hydraulic structure, cannot be adequately modelled, as it requires two boundary conditions at the upstream end. However, for cases of supercritical flow on steep bottom slopes the NewC scheme reproduces the normal depth corresponding to the given discharge, bottom slope, cross-sectional geometry and roughness.

The scheme can deal with hydraulic jumps at the breaks between steep and mild bottom slopes. In such cases, the scheme reproduces two normal depths and not the sequent depths. Consequently the loss of energy in a hydraulic jump is not modelled correctly but depends on the difference between the two bottom slopes.

Supercritical flows with a Froude number of less than 1.5 do not show any signs of noise. Flows with a higher Froude number start to show some wiggles that might grow into instability. To alleviate this problem the authors suggest the introduction of some numerical diffusion by increasing θ to 0.7.

Conclusions

The scheme presented in this paper is capable of modelling sub-, super- and trans-critical flow conditions in a unified way. To achieve that, no changes to the governing equations are required, which represents a major improvement on the schemes used in engineering applications currently. At the same time the solution based on this scheme maintains the same level of efficiency as the standardly used finite difference schemes. Although the overall solution algorithm for the new scheme differs from the algorithms currently used, it can easily be incorporated into algorithms for the solution of flows in free-surface networks.

List of symbols

x	spatial independent variable (m)
t	time independent variable (s)
y	water depth (m)
Q	discharge (m^3/s)
h	water level (m above sea level)
b_s	storage width (m)
H_k^n	Fourier coefficients for h
ξ_k^n	Fourier coefficients for Q
k	wave number (in Fourier series)
l	computational domain length (m)
α	dimensionless wave number,
A	cross-sectional area (m^2)
β	Boussinesq coefficient
g	gravity acceleration (m^2/s)
K	conveyance (m^3/s)
Cr	Courant number
Fr	Froude number
C	Chezy coefficient ($m^{1/2}$/s)

Indices

j	space (subscript)
n	time (superscript)

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