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A new algorithm for a robust solution of the fully dynamic Saint–Venant equations

Un nouvel algorithme pour une solution robuste des équations complètes de Saint–Venant

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ABSTRACT

A new procedure for the numerical solution of the fully dynamic shallow water equations is presented. The procedure is a fractional step methodology where the original system is split into two sequential ones. The first system differs from the original one because of the head gradient term, that is treated as constant and equal to the value computed at the end of the previous time step. The solution of this system, called kinematic, is computed in each element using a spatial zero order approximation for both the heads and the flow rates by means of integration of single ODEs. The second system is called diffusive, contains in the momentum equations only the complementary terms and can be easily solved using implicit methods. The major advantages of the methodology are: (1) it guarantees mass conservation; (2) it shows unconditional stability with respect to the Courant number; (3) it can be applied to initially dry domains; (4) it can be applied to closed conduits without the help of the Preissman approximation.

RÉSUMÉ

On présente une nouvelle methodologie pour la solution numérique des équations complètes des écoulements à surface libre. La méthode est une approache à intervalles fractionnés où le système original d'équations est divisé en deux systèmes sèquentiels. Le premier diffère de celui d'origine à cause du gradient de pression, qui est traité comune une costante pareille à celle calculée à la fin de l'intervalle précedent. La solution de ce système que l'on appelle de convection est calculée en chaque élément en utilisant une approssimation spatial de order zéro soit pour les pressions soit pour les débits à travers l'integration de chaque équation différentielle ordinaire qui a comune inconnue la profondeur de l'eau. La deuxième système est nommé de dispersion et il a dans le momentum équation seulement les termes complémentaires et il peut être facilement résolu en utilisant des méthodes implicites. Les plus grands avantages de cette méthodologie sont: (1) elle garantie la conservation de la masse; (2) elle montre de la stabilité inconditionelle par rapport au numéro de Courant; (3) elle peut être appliquée à des surfaces initialment sechès; (4) elle peut être appliquée àdes conduite d'eau sans l'aide de l'approximation de Preissman.

Keywords: Shallow water; flow routing; explicit methods.

1 Introduction

Two broad types of algorithms used for the numerical solution of the Saint–Venant (SV) equations are the implicit and the explicit ones. The implicit ones (Katopodes, 1984; Garcia-Navarro *et al*., 1994; Delis *et al*., 2000) solve the nodal variables (water levels and flow rates) at the unknown time level by solving large non-linear algebraic systems. Assuming appropriate finite difference schemes or weighting functions they allow Courant numbers larger than one, but the size of the time step is limited by the number of the required Newton–Raphson iterations, as well as by the artificial diffusion induced in the elements with very small Courant number. The explicit ones (Toro, 1992; Zoppou and Roberts, 1999a,b) compute the element variables one element after the other with the solution of one homogeneous and one ordinary differential equation. In spite of relevant advances that have been made for an exact solution of these equations, some "critical" conditions remain where the solution easily becomes unstable. The major weakness of the explicit methods, that have in these last years shown a better capability of dealing with these "critical" conditions, is that the time step size is conditioned to the Courant number or, more generally, to the velocity and the wave propagation celerity. This limitation becomes very serious especially in two cases. The first case is the existence of a limited area in the model domain, where the ground slope is steep, much more than the small values allowed by the shallow water Saint–Venant hypothesis. In this case the elements used in the numerical discretization are very small and the water velocities

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can be very large, along with the corresponding Courant number; this requires the choice of a very small time step, providing a large computational burden, as well as possible numerical diffusion in other parts of the domain. The second case is the transition from partial to full section in channels with closed sections. Assuming incompressible water, the wave celerity is infinite in a pressurized section and the same algorithm used to solve the free surface flow part of the domain cannot be used to solve the pressurized part. To avoid this inconvenience almost all the existing models use the Preissman approximation (Sjoberg, 1976), that is an approximation of the real, closed section with an open section displaying a very small top width, called the Preissman slot. This small width allows a finite value of the wave celerity and the use of the free surface flow model in all the parts of the computational domain. The counterbalance to this advantage is that the Preissman slot produces either a large wave celerity (and a corresponding time step limitation), if too small, or an evident storage effect (with flow rate peak reduction) if too large.

In the following, a new algorithm called DORA is applied for the unconditional solution of the SV equations. The consistency and the unconditional stability of the algorithm with respect to the size of the time step have been proven with the Fourier analysis for the groundwater transport problem (Tucciarelli and Fedele, 2000) only in the linear, 1D case. Extensive use of the same algorithm has also empirically shown these properties for both the 1D and the 2D diffusive form of the SV equations (Noto and Tucciarelli, 2001; Tucciarelli and Termini, 2000a). These early applications of the algorithm to the solution of the fully dynamic problem also suggest the same, good behavior.

2 The DORA algorithm

The fully dynamic SV equations can be written in the form:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial q}{\partial x} = Q, \tag{1}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{\sigma}\right) + g\sigma\frac{\partial H}{\partial x} + \frac{gn^2q^2}{\sigma\Re^{4/3}} = 0, \tag{2}$$

where σ is the flow section, q is the stream flow rate, h is the water depth, n is the Manning coefficient, H is the total water level (H is equal to h plus the bed elevation z), $\Re$ is the hydraulic radius and Q is the entering flow rate per unit length. Using a fractional step approach, Eqs. (1)–(2) are solved by means of the sequential solution of the following two systems of partial differential equations:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial q}{\partial x} = Q, \tag{3}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{\sigma}\right) + g\sigma\frac{\partial H^k}{\partial x} + \frac{gn^2q^2}{\sigma\Re^{4/3}} = 0, \tag{4}$$

where k is the index of the known time level, and

$$\frac{\partial \sigma}{\partial t} + \frac{\partial q}{\partial x} = \frac{\partial q^{k+1/2}}{\partial x} \tag{5}$$

$$\frac{\partial q}{\partial t} + g\sigma^{k+1/2}\frac{\partial H}{\partial x} = g\sigma^{k+1/2}\frac{\partial H^k}{\partial x} \tag{6}$$

where the gradients at the known time level k are treated as constant in Eqs. (3)–(4) and the variables with index $k+1/2$ are assumed to be known in Eqs. (5)–(6) from the previous solution of Eqs. (3)–(4). After simple transformations, we can cast systems Eqs. (3)–(4) and (5)–(6) respectively in the form:

$$\begin{pmatrix}\frac{\partial h}{\partial t}\\ \frac{\partial q}{\partial t}\end{pmatrix} + A_1\begin{pmatrix}\frac{\partial h}{\partial x}\\ \frac{\partial q}{\partial x}\end{pmatrix} = B_1 \text{ and } \begin{pmatrix}\frac{\partial H}{\partial t}\\ \frac{\partial q}{\partial t}\end{pmatrix} + A_2\begin{pmatrix}\frac{\partial H}{\partial x}\\ \frac{\partial q}{\partial x}\end{pmatrix} = B_2 \tag{7}$$

$$A_1 = \begin{pmatrix}0 & \frac{1}{L}\\ -V^2L & 2V\end{pmatrix}, \qquad A_2 = \begin{pmatrix}0 & \frac{1}{L}\\ g\sigma & 0\end{pmatrix},$$

where L is the section top width, A_1 has two coincident eigenvalues equal to the vertically averaged velocity V and A_2 has eigenvalues equal to $\pm\sqrt{g\bar{h}}$, where $\bar{h}$ is the average depth of the flow section. In the following, the system of Eqs. (3)–(4) is solved using a spatial zero-order approximation of both the water depth and the flow rate in each element between two nodes of the mesh and the system of Eqs. (5)–(6) is solved using a fully implicit finite difference algorithm.

2.1 Solution of the kinematic problem

Equations (3)–(4) are integrated in space in each element, where a constant value of q and h is assumed. This leads, in each element j, to the following couple of ordinary differential equations:

$$\frac{dh_j}{dt} = \frac{1}{\Delta x L_j}\left(\bar{q}_{ej}^k - q_j + Q_j\Delta x\right), \tag{8}$$

$$\frac{dq_j}{dt} = \frac{1}{\Delta x}\left(\bar{m}_{ej}^k - \frac{q_j^2}{\sigma_j}\right) - g\sigma_j\frac{H_{j+1}^k - H_j^k}{\Delta x} - g\frac{n^2q_j^2}{\sigma_j\Re_j^{4/3}}, \tag{9}$$

where L_j is the water section width, σ_j and $\Re_j$ are assumed to be functions of h_j (for sake of simplicity, not of x) and $\bar{q}_{ej}^k$ and $\bar{m}_{ej}^k$ are the entering flow rate and momentum along the Δt time step.

To solve Eqs. (8)–(9) for element j, $\bar{q}_{ej}^k$ and $\bar{m}_{ej}^k$ have to be known. This implies a need to order the elements according to the flow direction, so that the flux entering in the element with solution order i comes from an element that have a smaller solution order, and to solve Eqs. (8)–(9) sequentially starting from the upstream elements and ending at the downstream elements. This ordering operation is performed at each time step, to allow the switch of the flow direction in each element. Equations (8)–(9) are solved with any required precision using commercial subroutines for “stiff” systems with solutions displaying rapidly decaying terms. Even if Eqs. (8)–(9) could be integrated as a system of two ODEs, this would leave the singularity corresponding to zero initial water depth. To avoid this, it is convenient to approximate the flow rate time derivative as:

$$\frac{dq_j}{dt} \approx \frac{q_j - q_{jb}}{\Delta t_{jk}}, \qquad \Delta t_{jk} = \min\left(\Delta t, \frac{\Delta x}{V_{jk}^{pre}}\right), \tag{10}$$

where the ratio in the second Eq. (10) is the time step corresponding to a Courant number equal to one, according to the predicted velocity V_{jk}^{pre}, and q_{jb} is the flow rate estimated at a time equal to $t - \Delta t_{jk}$. After substitution, the flow rate is related to the water depth by the relationship:

$$q_j = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad b = \frac{d}{\Delta t_{jk}},$$
$$c = d\left(g\sigma_j \frac{H_{j+1}^k - H_j^k}{\Delta x} - \frac{\bar{m}_{ej}^k}{\Delta x} - \frac{q_{jb}}{\Delta t_{jk}}\right),$$
$$d = \frac{\Delta x \sigma_j \Re_j^{4/3}}{\Re_j^{4/3} + n^2 g \Delta x}, \tag{11}$$

where the singularity, for $h_j = 0$, is missing.

The leaving flow rate and momentum are the entering values for the next element and they can be numerically estimated along the running time step as functions of the solution of h_j, q_j. To improve the speed of the ODE solver their numerical estimation must be approximated with two simple polynomial functions $\bar{q}_{uj}^k$ and $\bar{m}_{uj}^k$. In a previous version of the algorithm (Tucciarelli and Termini, 2000b) these functions were constant values. This provides some artificial numerical diffusion at large Courant numbers. In the new version, second order polynomial functions are used. The corresponding six coefficients are estimated to satisfy the following conditions: (1) the initial values of the two functions are respectively equal to the flow rate and to the momentum computed at the end of the previous time step, (2) the final values of the two functions are equal to the flow rate and to the momentum computed at the end of the running time step, (3) the average (in time) values of the two functions are equal to the average flow rate and momentum computed along the running time step. The first two conditions are satisfied by assuming:

$$\bar{q}_{uj}^k(0) = a_{j1}^q = q_j^k, \qquad \bar{m}_{uj}^k(0) = a_{j1}^m = m_j^k, \tag{12}$$

$$\bar{q}_{uj}^k(\Delta t) = a_{j1}^q + a_{j2}^q \Delta t + a_{j3}^q \Delta t^2 = q_j^{k+1/2}, \tag{13}$$

$$\bar{m}_{uj}^k(\Delta t) = a_{j1}^m + a_{j2}^m \Delta t + a_{j3}^m \Delta t^2 = m_j^{k+1/2}, \tag{14}$$

where the time within brackets is relative to the known time level. Because the solution is not very sensitive to the momentum for small water depths, where the kinematic approximation holds, the momentum can be estimated in Eqs. (12) and (14) as:

$$m_j^k = \frac{\left(q_j^k\right)^2}{\max\left(\sigma_{\min}, \sigma_j^k\right)}, \tag{15}$$

where $\sigma_{\min}$ is the section area corresponding to a very small water depth (10^{-6} m).

The third condition is satisfied for the flow rate by applying the mass conservation at element j along time step Δt:

$$a_{j-1,1}^q + a_{j-1,2}^q \frac{\Delta t}{2} + a_{j-1,3}^q \frac{\Delta t^2}{3}$$
$$= a_{j1}^q + a_{j2}^q \frac{\Delta t}{2} + a_{j3}^q \frac{\Delta t^2}{3}$$
$$+ \frac{L_j^k + L_j^{k+1/2}}{2}\left(h_j^{k+1/2} - h_j^k\right)\Delta x / \Delta t, \tag{16}$$

where $j - 1$ is the index of the upstream element. Due to the non-linearity of the momentum equation, the exact momentum conservation is not possible. The average momentum $\bar{\bar{m}}_{uj}^k$ is estimated by integrating Eq. (9):

$$\bar{\bar{m}}_{uj}^k = \frac{1}{\Delta t}\int_0^{\Delta t}\left[\bar{m}_{ej}^k - \left(\frac{n^2 g q_j^2}{\sigma_j \Re_j^{4/3}} + g\sigma_j \frac{H_{j+1}^k - H_j^k}{\Delta x} + \frac{dq_j}{dt}\right)\Delta x\right] dt. \tag{17}$$

For a more accurate integration it is convenient to use a time centered derivative approximation instead of the fully implicit one applied in the relationship (11) to solve the ODE for the unknown, h. Equation (17) leads to the last condition:

$$a_{j1}^m + a_{j2}^m \frac{\Delta t}{2} + a_{j3}^m \frac{\Delta t^2}{3} = \bar{\bar{m}}_{uj}^k. \tag{18}$$

The entering flow rate and momentum in the next element $j + 1$ are given by:

$$\bar{q}_{e(j+1)}^k = \bar{q}_{uj}^k, \qquad \bar{m}_{e(j+1)}^k = \bar{m}_{uj}^k. \tag{19}$$

2.2 Solution of the diffusive problem

Equations (5)–(6) are solved using an implicit finite difference procedure. According to Eq. (6), the flow rate is given by:

$$q_j^{k+1} = -\Delta t g \sigma_j^{k+1/2} \frac{H_{j+1}^{k+1} - H_j^{k+1}}{\Delta x}$$
$$+ \Delta t g \sigma_j^{k+1/2} \frac{H_{j+1}^k - H_j^k}{\Delta x} + q_j^{k+1/2}. \tag{20}$$

Equation (20) can be written in the form:

$$q_j^{k+1} = -disp_j^{k+1/2} \frac{H_{j+1}^{k+1} - H_j^{k+1}}{\Delta x} + conv_j^{k+1/2}, \tag{21}$$
$$disp_j^{k+1/2} = -\Delta t g \sigma_k^{k+1/2},$$
$$conv_j^{k+1/2} = \Delta t g \sigma_j^{k+1/2} \frac{H_{j+1}^k - H_j^k}{\Delta x} + q_j^{k+1/2}.$$

Using a fully implicit approximation of the space flow rate derivatives and substituting the right hand side of Eq. (21) in Eq. (5), one gets, for $j = 1, \ldots, N$

$$\frac{H_j^{k+1} - H_j^{k+1/2}}{\Delta t} cap_j^{k+1/2} + \delta_j^N disp_j^{k+1/2} \frac{H_j^{k+1} - H_{j+1}^{k+1}}{\Delta x}$$
$$- \delta_j^1 disp_{j-1}^{k+1/2} \frac{H_{j-1}^{k+1} - H_j^{k+1}}{\Delta x}$$
$$= \delta_j^1 conv_{j-1}^{k+1/2} - \delta_j^N conv_j^{k+1/2}$$
$$- \delta_j^1 q_{j-1}^{k+1/2} + \delta_j^N q_j^{k+1/2} + \left(1 - \delta_j^N\right)F, \tag{22}$$

$$cap_j^{k+1/2} = \Delta x L_j^{k+1/2}, \tag{23}$$

where δ_j^1 is equal to one if j is greater than one and equal to zero if not, δ_j^N is equal to one if j is smaller than N and equal to zero if not, all the elements are ordered from 1 to N along the stream direction and F depends on the downstream boundary condition. The matrix associated to the linear system (22) is symmetric and positive definite and allows a fast solution of the problem.

3 Boundary conditions

3.1 Boundary conditions of the kinematic problem

Both the eigenvalues of the kinematic problem are positive. In this case two upstream boundary conditions are given by assigning both flow rate and momentum. If the flow, as computed at the end of the previous time step, is subcritical, the water depth computed as the solution of the previous diffusive problem is used to estimate the entering momentum. The downstream conditions are missing, but the water level gradients of the kinematic problem are computed at the end of the previous time step in the forward direction using the water level of the following element. This implies the need, at the last downstream element, to replace the momentum equation with the equation of the characteristic line:

$$\frac{dx}{dt} = \frac{1}{V}, \tag{24}$$

where the velocity V is approximated as constant along the element. The foot of the characteristic line ending in the space-time plane at the $j\Delta x$, t point falls at the boundary with the previous element $j-1$ if the Courant number in element j is greater than one (case 1, Figure 1). In this case to estimate the velocity in Eq. (24) a linear variation is assumed between the velocities in element $j-1$ at time levels k and $k+1/2$. If the Courant number is smaller than one, the velocity is assumed constant and equal to the value computed at time level k (case 2, Figure 1).

3.2 Boundary conditions of the diffusive problem

The upstream condition of the diffusive problem is always given by zero entering flux. The downstream condition is given by the flow rate computed with the solution of the kinematic problem if the flow is supercritical and not pressurized ($F = q_N^{k+1/2}$). If the flow is supercritical but pressurized, a water level equal to the level of the top of the conduit has to be assigned. If the flow is subcritical, the water depth can be assigned, as well as the free-fall condition:

$$q = \sigma(h)\sqrt{g\bar{h}}, \tag{25}$$

where $\bar{h}$ is the average depth of the section. If the flow is pressurized and subcritical the condition:

$$h = \min(h_c, h_{\max}) \tag{26}$$

is assigned, where h_c is the critical depth corresponding to the flow rate $q_N^{k+1/2}$ and $h_{\max}$ is the maximum depth of the conduit.

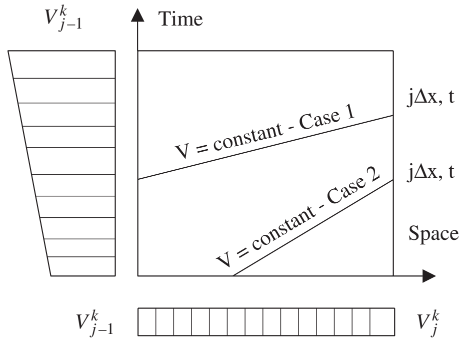

Figure 1 Representation of the characteristic lines inside the downstream boundary element.

4 Bore computation

When a bore arises in the stream flow, the water level discontinuity provides an infinite value of the spatial gradient in Eq. (4). The bore computation can be handled in the solution of the kinematic problem in the following way:

Define for each element j an upstream bore water depth h_{bj} and a distance d_{bj} of the discontinuity from the upstream end of the element (see Figure 2). Assume that the flow rate q_{bj} of the upstream side of the bore is equal, at the end of the running time step, to the flux $\bar{q}_{ej}^k(\Delta t)$ entering in the element at the same time level. Application of the mass and momentum conservation law around the bore leads to the system (Chow, 1959):

$$c_{bj} = \sqrt{\frac{g h_{bj}}{2h_{vj}}(h_{bj} + h_{vj})} + V_{vj}, \quad c_{bj} = \frac{q_{bj} - q_{vj}}{\sigma_{bj} - \sigma_{vj}}, \tag{27}$$

where c_{bj} is the bore celerity. A rectangular section has been assumed for sake of simplicity. After the unknowns h_{bj} and c_{bj} are found, two possibilities exist. In the first case the Froud number associated with the relative velocities $V_{bj} - c_{bj}$, $V_{vj} - c_{bj}$ of the bore and of the element are both smaller or greater than one; in this case the bore does not exist and the solution of the system (8)–(9) is found. In the second case the bore exists and the final distance of the bore is found by applying the integral form of the mass conservation law at the element volume. Observe that the bore celerity at the end of the running time step is different from the average one.

In the element preceding the element where a bore exists the kinematic downstream boundary condition (24) is applied. If the new distance between the bore and the upstream end of element j is greater than the element size, the water depth of the element is replaced with h_b and the mass conservation law provides again the average (in time) flux leaving element j for the next one.

Diffusion in the element $j-1$ preceding the element j where the bore exists is neglected. This avoids the need of estimating

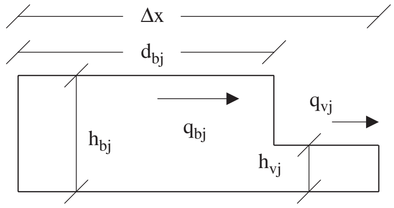

Figure 2 Representation of the water depth inside an element where a bore is located.

the water level gradient between element $j-1$ and j in (22) and it is accomplished by setting in the same equation:

$$disp_{j-1}^{k+1/2} = 0, \quad conv_{j-1}^{k+1/2} = q_{j-1}^{k+1/2}. \tag{28}$$

Observe that the proposed procedure allows an exact location of the bore when the bore celerity direction is the same flow direction. In these cases, like in two of the three tests proposed by Katopodes (1984), the methodology provides the exact solution with the use of any Courant number. If the bore celerity direction is opposite to the flow direction, it is possible that its location moves, along the time step, from a given element j to a new element that has already been solved. This implies the need of neglecting the bore existence in element j; due to the conservative form of Eqs. (3)–(4) this has not effect on the stability of the solution.

5 Transition from partial to full section without the use of the Preissman approximation

One of the most useful capabilities of the DORA scheme is its ability to simulate flows in a channel with a closed section, and the transition from free surface to pressurized flow without any approximation of the section geometry, like in the Preissman scheme (Preissman, 1961). This capability is due to the unconditional stability of the solution of the diffusive problem, that can handle instantaneous transmission of pressure and velocity changes arising in the pressurized part of the channel, when water compressibility is neglected. Mass conservation can be guaranteed in the solution of the kinematic part of the problem, for each element j, by using the following procedure:

(1) Instead of solving the kinematic problem in element j at time level $k+1/2$, if the total water level exceeds at time level k the maximum water depth $h_{\max}$, set:

$$\bar{q}_{uj}^{k+1/2} = \bar{q}_{ej}^{k+1/2}, \qquad \bar{m}_{uj}^{k+1/2} = \bar{m}_{ej}^{k+1/2}, \tag{29}$$

and continue to solve the kinematic problem of the downstream element;

(2) Otherwise, stop the solution of the ODEs (8)–(9) in element j when the computed water depth h_j reaches the maximum water depth $h_{\max}$;

(3) If the maximum depth is reached, solve the same ODE from the initial water depth to the $h_{\max}$ value, in the time unknown. This provides the time $t_{\max}$ required for the upstream flow to fill up the element volume, according to the zero order spatial approximation and to the fixed water level gradient. Compute and save the exceeding volume that cannot be transferred along the time step k through element j to the downstream part of the channel, according to the fixed water level gradient. This volume is given by:

$$Vol_j^k = \int_{t_{\max}}^{\Delta t} \bar{q}_{ej}^k(t)\,dt - \bar{q}_{ej}^k(t_{\max})(\Delta t - t_{\max}). \tag{30}$$

A constant value is assumed in this case for the leaving flux and momentum:

$$\bar{q}_{uj}^k = \frac{1}{\Delta t}\int_0^{\Delta t} \bar{q}_{ej}^k\,dt - \left(h_{\max} - h_j^k\right)\left(\frac{L_j^k + L_{\max}}{2\Delta t}\right) - \frac{Vol_j^k}{\Delta t}, \tag{31}$$

$$\bar{m}_{uj}^k = \bar{\bar{m}}_{uj}^k, \tag{32}$$

where the right hand side in Eq. (32) is given by Eq. (17). In the solution of the diffusive problem, add at element j the flow rate given by the last term in Eq. (31). In the same element, as well as in all the elements where Eqs. (29) have been applied, change Eq. (21) with the following momentum equation for pressurized flow:

$$\frac{q_j^{k+1} - q_j^k}{\Delta t} = -g\sigma_j^{k+1/2}\frac{H_{j+1}^{k+1} - H_j^{k+1}}{\Delta x} - \frac{gn^2 q_j^{k+1} q_j^{k+1/2}}{\sigma_j^{k+1/2}\left(\Re_j^{k+1/2}\right)^{4/3}}, \tag{33}$$

coupled with the continuity equation:

$$q_j^{k+1} - q_{j-1}^{k+1} = Q_j \Delta x + \frac{Vol_j}{\Delta t}. \tag{34}$$

Use of Eqs. (33)–(34) instead of (5)–(20) results in the same Eq. (21), where the coefficients are changed according to:

$$disp_j^{k+1/2} = \frac{g\Delta t\left(\sigma_j^{k+1/2}\right)^2\left(\Re_j^{k+1/2}\right)^{4/3}}{\sigma_j^{k+1/2}\left(\Re_j^{k+1/2}\right)^{4/3} + n^2 g\Delta t q_j^{k+1/2}},$$

$$conv_j^{k+1/2} = \frac{q_j^k \sigma_j^{k+1/2}\left(\Re_j^{k+1/2}\right)^{4/3}}{\sigma_j^{k+1/2}\left(\Re_j^{k+1/2}\right)^{4/3} + n^2 g\Delta t q_j^{k+1/2}},$$

$$cap_j^{k+1/2} = 0. \tag{35}$$

6 Test cases

In all the tests the channel section is assumed to be infinitely large, the flow rates are given per meter and the bed is assumed to be initially dry. The upstream boundary condition is given by the flow rate and, in the case of supercritical, not pressurized flow, by the uniform flow condition. The Manning coefficient is $0.05\ \mathrm{s\,m^{1/3}}$.

6.1 Flow simulations with different time steps

The algorithm is first applied to the propagation of an inflow hydrograph using two different time steps, equal to 1 and 20 s. The slope of the channel is 0.01 and the length is 1000 m. See in Figure 3 three lines, corresponding to the entering hydrograph and to the two leaving ones. The maximum Courant number obtained using the larger time step is 12.45. The peak is reduced in this second case, due to the numerical diffusion, only from 4.599 to 4.588 $\mathrm{m^3/s}$.

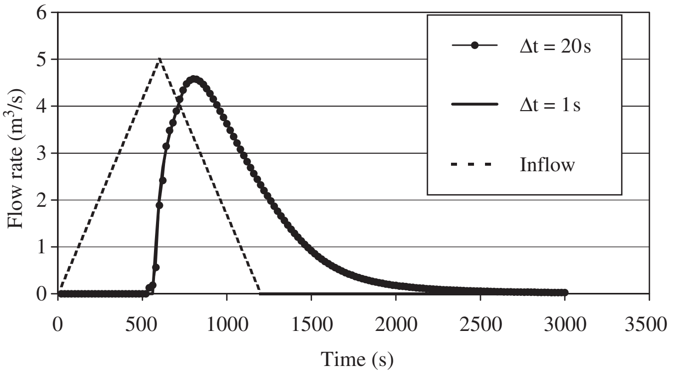

Figure 3 Simulations with different time steps.

6.2 Dam-break with finite downstream water depth

The results of the algorithm are compared with the analytical solution of the following dam-break problem: an horizontal, infinitely large and frictionless channel is divided in two parts by a vertical gauge. The water depth is equal to 100 m in the upstream part of the channel and to 1 m in the downstream part (Figure 4). Assuming an infinite distance of the boundaries with respect to the gauge location, the water depth and the water velocities are computed at a time $t = 9.9$ s. The solution is computed using a time step of 0.1 s and a spatial step of 10 m.

The analytical solution of this problem can be found in Louaked and Hanich (1998). A comparison of the efficiency of 11 different explicit models, according to this specific example and to specific norms, can be found in Zoppou and Roberts (1999b). The results of the proposed model provide a norm of 0.0086 for the head error N_h and of 0.051 for the velocity error N_v, corresponding to an average ranking in the "performance" list. The two norms are respectively defined as:

$$N_h = \frac{\sum_{j=1}^{N} \left| h_j^{99} - h_{jt}^{99} \right|}{\sum_{j=1}^{N} \left| h_{jt}^{99} \right|}, \qquad N_v = \frac{\sum_{j=1}^{N} \left| V_j^{99} - V_{jt}^{99} \right|}{\sum_{j=1}^{N} \left| V_{jt}^{99} \right|}, \tag{36}$$

where h_{jt}^{99} and V_{jt}^{99} are the analytical solution of the head and of the velocity at the center of element j at the end of the 99th time step. The minimum N_h value obtained by one of the 11 models is 0.0063 and the minimum N_v value (given by an other one) is 0.029. The better performance obtained in the head estimation with respect to the velocity estimation can be explained with the use of flow rates as unknowns, instead of velocities. In Figure 4 the computed water depths show the good agreement with the analytical solution of the problem. Observe that the norms chosen by the authors are very sensitive to the bore location.

6.3 Dam-break with zero downstream water depth

The analytical solution of the dam-break problem when the downstream water depth is zero is different from the previous one and is given by the equation of a parabola with vertical axis and vertex on the channel bed (Chow, 1959). The results of the same previous problem, run with zero downstream water depth, are compared in Figure 5 with the analytical solution.

6.4 Conduit with closed section

The algorithm is finally applied to a the case of a rectangular closed conduit one meter high, where only the bottom is assumed to provide friction. The Manning coefficient is 0.05 s $m^{1/3}$ and the length is 1000 m. An entering triangular hydrograph is assigned and the leaving one is computed for the two cases of slope equal to 0.005 and 0.001 (Figure 6). In the pressurized part of the channel the resistance does not grow along with the piezometric level. This implies that the rising limb of the inflow hydrograph produces a clear sharp front effect and the outflow hydrograph quickly moves from a flow rate equal to zero to a flow rate exactly equal to the inflow value. Observe that the use of the Preissman approximation would not allow to exactly match the

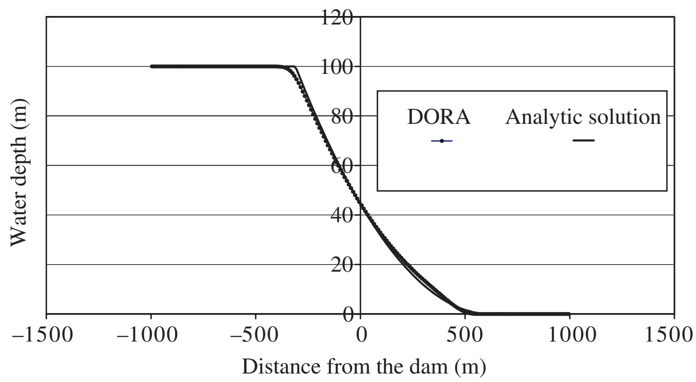

Figure 5 Dam break – zero downstream water depth.

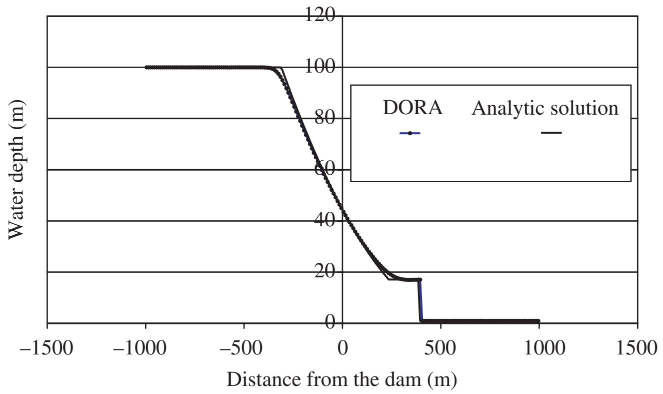

Figure 4 Dam break – finite downstream depth.

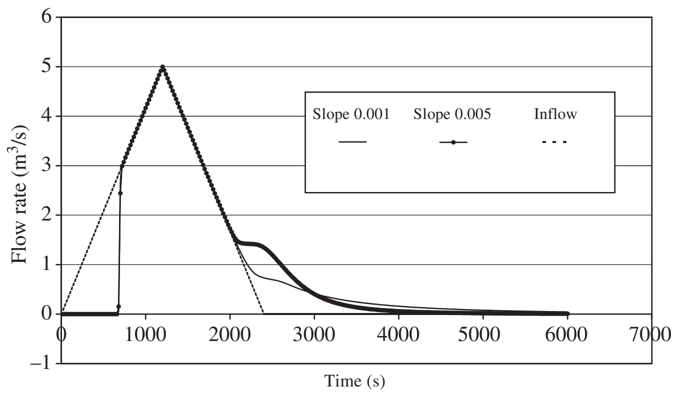

Figure 6 Closed conduit.

inflow hydrograph, due to the finite wave celerity that has to be ensured with the use of the Preissman slot.

When the flow rate decreases below a minimum value, the critical depth corresponding to the channel flow rate becomes smaller than the channel height and part of the stored water volume is released, producing a net limitation of the flow rate reduction velocity. This limitation is more pronounced for the larger value of the channel slope, because in that case the release of the stored water is faster. Observe that the integral of the entering and of the leaving hydrograph are the same and the total volume conservation is exactly satisfied.

7 Conclusion

The 1D fully dynamic Saint–Venant equations have been solved with the use of a new algorithm, called DORA, previously applied to the solution of the diffusive model. The most important merit of the algorithm is its ability to handle without singularities the zero water depth condition and to solve the convective part of the equations using Courant numbers much larger than one. This solution is explicit, in the sense that it requires the solution of a sequence of single ODEs. To overcome the so-called Courant-Friederichs-Levy condition (Fletcher, 1991) the method uses the information coming from the resolved upstream element. The counterbalance of this advantage is the need of ordering the element, at each time step, according to the stream direction. One important consequence of this property is the ability to solve the transition from free surface to pressurized flow without the help of the Preissman approximation.

Notation

q = channel flow rate
Q = entering flux per unit length
h = water depth
$\bar{h}$ = average depth corresponding to h
L = water section width
σ = channel section corresponding to h
$\mathfrak{R}$ = hydraulic radius corresponding to h
z = ground elevation
H = total water level ($H = z + h$)
g = gravity force per unit mass
n = Manning coefficient
N = number of elements
m = momentum per unit time
$\bar{m}, \bar{q}$ = approximated momentum and flux from time t^k to time $t^k + \Delta t$ (quadratic function of time)
Δt = time step in the diffusive problem
a_{j1}, a_{j2}, a_{j3} = coefficients of the second order approximation
$\bar{\bar{m}}$ = time average momentum
c = bore celerity
V = space average velocity
Vol_j^k = volume exceeding the transfer capacity of element j along time level
disp, *conv* = coefficients of the linear system solving the diffusive problem
δ_j^k = equal to 1 in any element different from k, equal to 0 in k
F = downstream flux boundary value

Subscripts

c = critical value
j = element index
k = known time level index
max = top level value in a closed conduit
min = value corresponding to a minimum water depth of 10^{-6} m
e = value entering the element
u = value leaving the element
b = upstream bore value
v = downstream bore value
N = index of the last downstream element

Superscripts

k = known time level
$k + 1/2$ = unknown time level in the kinematic problem
$k + 1$ = unknown time level in the diffusive problem
pre = predicted value
m = coefficient of the momentum second order approximation
q = coefficient of the flow rate second order approximation

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