Conservation-form equations of unsteady open-channel flow

Formes conservatives des équations pour un écoulement non permanent en canal

C. LAI, *Hydrologist (retired), U.S. Geological Survey, and Visiting Professor, Hydrotech Research Institute, National Taiwan University, Taipei, Taiwan*

R.A. BALTZER, *Hydrologist (retired), U.S. Geological Survey, National Center MS 430, Reston, VA, 20192*

R.W. SCHAFFRANEK, *Hydrologist, U.S. Geological Survey, National Center MS 430, Reston, VA, 20192*

ABSTRACT

The unsteady open-channel flow equations are typically expressed in a variety of forms due to the imposition of differing assumptions, use of varied dependent variables, and inclusion of different source/sink terms. Questions often arise as to whether a particular equation set is expressed in a form consistent with the conservation-law definition. The concept of conservation form is developed to clarify the meaning mathematically. Six sets of unsteady-flow equations typically used in engineering practice are presented and their conservation properties are identified and discussed. Results of the theoretical development and analysis of the equations are substantiated in a set of numerical experiments conducted using alternate equation forms. Findings of these analytical and numerical efforts demonstrate that the choice of dependent variable is the fundamental factor determining the nature of the conservation properties of any particular equation form.

RÉSUMÉ

Les équations instationnaires des écoulements en canaux peuvent s'écrire sous diverses formes selon les différentes hypothèses à imposer, l'usage de différentes variables dépendantes, et la prise en compte de différents termes source/puits. La question qui se pose souvent est de savoir si un système d'équation, exprimé sous une forme donnée, est consistant avec la définition des lois de conservation. Le concept de forme conservative est développé (ici) pour en clarifier mathématiquement le sens. On présente six systèmes d'équations instationnaires, typiquement utilisées en ingénierie, dont on identifie et discute les propriétés de conservation. Les résultats du développement et de l'analyse théoriques des équations sont justifiés par un ensemble d'expériences numériques menées en utilisant les diverses formes des équations. Le verdict de ces efforts analytiques et numériques est que le choix des variables dépendantes est le facteur fondamental qui va déterminer la nature des propriétés de conservation de chaque forme particulière d'équation.

Introduction

The beginning of the modern study of unsteady flow in open channels can be traced to the latter half of the nineteenth century when the partial differential equations of continuity and motion were introduced by the French engineer de Saint Venant (1871). Subsequently, the equations have been subjected to a variety of modifications, e.g., Boussinesq (1877). Analytical solutions to these nonlinear equations have not been found except for cases of greatly simplified or idealized flow, or by resorting to graphical methods, e.g., Massau (1900). Therefore, it is of great interest and need to identify equation forms for numerical solution that preserve the basic mathematical and physical attributes of the underlying hydraulic principles.

The principles of mass, momentum, and energy conservation are used to describe the physics of fluid motion. Typically, the fundamental equations governing unsteady flow in open channels are premised on conservation of mass and momentum in a bounded system. A variety of unsteady-flow equation forms can be found and are typically required to accurately depict the many forcing mechanisms that generate the spectrum of dynamic components of flows in open channels. Moreover, frequently other terms must be added to the more fundamental equation forms in order to produce a widely useful numerical simulation model. Questions often arise, however, as to whether the resultant equations are expressed in, so-called, *conservation form.* The nature and meaning of conservation-law form have been addressed in the literature (cf. Abbott, 1979; Cunge et al., 1980; Liggett, 1975) and, in general, these analyses are founded on the treatise of the system of conservation laws presented by Lax (1957). Upon formulation of an equation set consistent with the conservation requirements for the intended applications, a variety of differential forms can be developed to affect a solution by following integration methods appropriate for initial-boundary-value problems, e.g., Richtmyer and Morton (1967), while also preserving their conservation-law form, e.g., LeVeque (1992).

In recognition of conflicting interpretations and uses of the term *conservation form,* preliminary review, analysis, and discussion of its meaning and significance are conducted prior to analyzing varied forms of the unsteady-flow equations. Moreover, in view of the need to encompass treatment of additional forces and conditions in unsteady open-channel flow models, the effects of modifications and extensions to fundamental forms of the equations are considered and treated. In this investigation, the writers (i) review and clarify classical (normally referred to as *conservation-law)* and hydraulic-engineering *conservation* forms of the unsteady flow equations, (ii) define forms of the unsteady flow equations for six dependent-variable combinations typically used in engineering practice, (iii) identify those dependent-variable combinations conforming to conservation-law form, (iv) discuss implications and consequences of modifications to the equations, and (v) conduct numerical experiments designed to demonstrate solution differences between equation forms.

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Theoretical considerations

The notion of conservation form has its roots in the mathematical treatment of the *flux* and *accumulation* of physical quantities, e.g., mass, momentum, energy, etc., associated with fluid flow in a control volume or conduit. To treat real fluids, which exhibit energy dissipation, flowing in real conduits, e.g., rivers and waterways, it is necessary to consider not only the net flux and accumulation of these various quantities, but also the addition or loss of quantities due to external sources or sinks and the effects of irreversible forces acting on the flow.

Fundamental form of conservation laws

The unsteady flow equations, or initial-value-problem equations, when expressed in one-(space) dimensional, conservation-law form (Lax and Wendroff, 1960; Richtmyer and Morton, 1967), generally are written:

$$\frac{\partial}{\partial t}\mathbf{U}+\frac{\partial}{\partial x}\mathbf{F}(\mathbf{U})=0 \tag{1}$$

in which **U** and **F(U)** are each a column vector of *N* components. Equation (1) states that the net flux of each quantity in **U** is exactly balanced by its time rate-of-change in a control volume. Thus, Eq. (1) is the so-called 'conservation' or 'divergence' form of these equations. The term divergence implies that all spatial derivatives appear purely as divergences in the mathematical sense (Cunge et al., 1980; Hirsch, 1988; Roache, 1998). Conservation form can be expressed, also, by writing Eq. (1) in integral form, $\oint_R [\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x]\, dR = 0$. The contour integral indicates that neither a gain nor loss of flux occurs within the finite space-time domain, *R*, prescribed by the bounding contour.

Two variant equations can be derived from Eq. (1). For hyperbolic-type equations, Eq. (1) can be expressed as

$$\frac{\partial}{\partial t}\mathbf{U}+\mathbf{A}(t,x,\mathbf{U})\frac{\partial}{\partial x}\mathbf{U}=0, \tag{2}$$

in which **A**, an *N* x *N* matrix, is a Jacobian, i.e., $\mathbf{A} = \partial\mathbf{F}/\partial\mathbf{U}$. Here, from **A**, or more specifically, from

$$|\mathbf{A}-\lambda\mathbf{I}|=0, \tag{3}$$

in which **I** is the identity matrix, *N* characteristics or (real and distinct) eigenvalues denoted by $\lambda_j = dx/dt$, can be found. By substituting the λ_j thus found into Eq. (2), the characteristics form of the equation results:

$$\frac{D\mathbf{U}}{Dt}=\frac{\partial\mathbf{U}}{\partial t}+\lambda_j\frac{\partial\mathbf{U}}{\partial x}=0 \tag{4}$$

wherein the total differential operator signifies

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{dx}{dt}\frac{\partial}{\partial x}=\frac{\partial}{\partial t}+\lambda_j\frac{\partial}{\partial x}. \tag{5}$$

Equation (4) has a specific physical meaning. It signifies that the flux quantity of interest (say, the *i*th component in **U**) remains constant as it moves along the characteristic λ_j, $j = 1, 2, \ldots, N$, or as it moves with velocity $dx/dt = \lambda_j$ in the *x*-direction [see Eq. (5)]. Thus, Eq. (4) has the same conservation attributes as Eq. (2) by virtue of Eq. (3).

If some external factors (e.g., a frictional force) act to upset the balance as stated above, or sources or sinks increase or decrease the flux quantity, the nonconservation form of equations (cf. Gabutti, 1983) results:

$$\frac{\partial}{\partial t}\mathbf{U}+\frac{\partial}{\partial x}\mathbf{F}(\mathbf{U})=\mathbf{G} \tag{6}$$

$$\frac{\partial}{\partial t}\mathbf{U}+\mathbf{A}\frac{\partial}{\partial x}\mathbf{U}=\mathbf{G} \tag{7}$$

$$\frac{D\mathbf{U}}{Dt}=\mathbf{G}_1 \tag{8}$$

in which $\mathbf{G} = \mathbf{G}(x,t,\mathbf{U})$ is a column vector of *N* components, representing the source or sink terms. Equation (8), for example, means that the quantity will no longer be conserved along the characteristic path; instead it is dynamically balanced by the amount $\mathbf{G}_1$.

In summary, Eqs. (1), (2), and (4) reflect conservation-law form in its strict sense, and they will be referred to as being the *basic* or *pure* conservation form. Equations (6), (7), and (8) reflect a more inclusive concept of conservation law, and thus, will be referred to as being the *extended* conservation form. For the purposes of this study, *both* forms will be referred to simply as conservation form.

Expanded forms of equations

In engineering practice, derivation of the unsteady open-channel flow equations typically uses the same flux-accumulation approach described above, but with the addition of source/sink terms. Usually, the principles of mass and momentum conservation are applied. The derivation yields two partial differential equations (PDEs) of hyperbolic type comprised of two dependent variables $\mathbf{U} = [\Phi,\Psi]^T$ and two independent variables, time, *t*, and longitudinal distance along the channel, *x*, (Lister, 1960; Lai, 1986),

$$\mathbf{M}\frac{\partial\mathbf{U}}{\partial t}+\mathbf{N}\frac{\partial\mathbf{U}}{\partial x}+\mathbf{E}=0 \tag{9}$$

in which **M** and **N** are equal to $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$, respectively, and $\mathbf{E} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is zero for the pure conservation form.

Here, all elements of **M**, **N**, and **E** are real. Equation set (9) is generally comprised of two first-order, quasi-linear PDEs for which the corresponding characteristic equations can be derived either by an algebraic or matrix approach (Lai, 1986).

The characteristic equations consist of two characteristics $\lambda_\pm$ defined as:

$$\left(\frac{dx}{dt}\right)_\pm = \lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad b^2 - 4ac > 0, \tag{10}$$

in which $a = m_{11}m_{22} - m_{12}m_{21} = |\mathbf{M}|$, $b = -m_{22}n_{11} + m_{12}n_{21} - m_{11}n_{22} + m_{21}n_{12}$, and $c = n_{11}n_{22} - n_{12}n_{21}$, and two compatibility equations associated with the characteristics

$$\frac{D\Phi}{Dt}_\pm + \Lambda_\pm \frac{D\Psi}{Dt}_\pm = G'_\pm \tag{11}$$

in which $\Lambda_\pm$ represents variable coefficients $\Lambda_\pm = \Lambda_\pm(\Phi,\Psi)$,

$G'_\pm$ represents the nonhomogeneous terms, and $\frac{D}{Dt}$ is as defined in Eq. (5).

Now, if **M** is factored out, Eq. (9) becomes

$$\mathbf{M}\left[\mathbf{I}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{K}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\right] = 0 \tag{12}$$

in which $\mathbf{K} = \mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{B} = \mathbf{M}^{-1}\mathbf{E}$. Because $|\mathbf{M}| \neq 0$, the solution of Eq. (9) is equivalent to solving

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{K}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} = 0 \tag{13}$$

which is in the form of Eq. (7). Thus, the characteristics, $\lambda_\pm$, can be found from

$$|\mathbf{K} - \lambda\mathbf{I}| = 0. \tag{14}$$

It is apparent that the form Eq. (13) can be obtained by pre-multiplying Eq. (9) by $\mathbf{M}^{-1}$, which is equivalent to performing linear combinations of the two component equations. The Eq. (13) form is sometimes called a *reduced* form in comparison with Eq. (9). It is not only interesting but also important to note that the two forms yield identical characteristic roots (or values), but *not* necessarily identical characteristic vectors. The proof can be shown as follows:

From Eq. (14), $\mathbf{K}\frac{\partial \mathbf{U}}{\partial x} = \lambda\frac{\partial \mathbf{U}}{\partial x}$ and Eq. (13) can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \lambda\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} = 0. \tag{15}$$

Pre-multiplying Eq. (15) by **M**, gives

$$\frac{\partial \mathbf{V}}{\partial t} + \lambda\frac{\partial \mathbf{V}}{\partial x} + \mathbf{E} = 0 \tag{16}$$

in which $\mathbf{V} = \int \mathbf{M}d\mathbf{U}$, $\frac{\partial \mathbf{V}}{\partial t} = \mathbf{M}\frac{\partial \mathbf{U}}{\partial t}$, and $\frac{\partial \mathbf{V}}{\partial x} = \mathbf{M}\frac{\partial \mathbf{U}}{\partial x}$.

Equation (16) is in a form of matrix equation

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{W}\frac{\partial \mathbf{V}}{\partial x} + \mathbf{E} = 0 \tag{17}$$

where

$$|\mathbf{W} - \lambda\mathbf{I}| = 0 \tag{18}$$

or

$$\mathbf{W}\frac{\partial \mathbf{V}}{\partial x} = \lambda\frac{\partial \mathbf{V}}{\partial x} \tag{19}$$

However, Eq. (17) is equal to Eq. (9); hence, Eqs. (13) and (9) have *identical* characteristic roots, λ, but generally *not identical* characteristic vectors, i.e., $\mathbf{V} \neq \mathbf{U}$, unless $\mathbf{M} = \mathbf{I}$.

Once the form of Eq. (13) for $\mathbf{U} = [\Phi_1,\Psi_1]^T$, or that of Eq. (17) for $\mathbf{V} = [\Phi_2,\Psi_2]^T$, is obtained, the solution form

$$\frac{D\mathbf{U}}{Dt} = -\mathbf{B}, \text{ or } \frac{D\mathbf{V}}{Dt} = -\mathbf{E},$$

each along the identical characteristic path, λ, can be written. Because any linear combination of such solutions is also a solution, the Eq. form (11), with (Φ,Ψ) specified as (Φ_1,Ψ_1) or (Φ_2,Ψ_2), can be obtained.

From the preceding arguments, therefore, it can be concluded that the solution of Eq. (9), the general unsteady-flow equation set, is the solution of an imbedded conservation-form equation set in the context of this paper, i.e., the equation set within the bracket of Eq. (12). If $\mathbf{M} = \mathbf{I}$, the original equation set is in conservation form directly. If $\mathbf{M} \neq \mathbf{I}$, the original equation set is a conservation-form equation set pre-multiplied by a factor **M**. A unique set of characteristics can be obtained in either case. In the former, the flux quantities conserved in the sense of Eq. (8) [Eq. (4), if $\mathbf{E} = 0$] are exactly those for which conservation was originally intended. In the latter, however, the pair of quantities conserved pertains only to the expression within the brackets, and in general, these are not the quantities for which conservation was originally sought. It is to be noted, here, that the original form of the latter equation set can always be reduced to the form in the brackets through pre-multiplication by $\mathbf{M}^{-1}$, and the form (in the brackets) thus obtained is called the *reduced* form, as identified previously. The physical interpretation of these statements is significant and of interest. When the equations for unsteady flow in open channels are derived based on conservation principles (e.g., mass and momentum conservation), a set of quasi-linear first-order hyperbolic-type PDEs, represented by Eq. (9), results. If the derived equation set has an **M** that is in the **M** = **I** form, then the quantities of concern (e.g., mass and momentum) are each conserved according to the form of Eq. (2)/(4), or to form Eq. (7)/(8) in the *extended* sense. On the other hand, if $\mathbf{M} \neq \mathbf{I}$, (say, due to the selection of a different set of dependent variables), the original pair of quantities (e.g., mass and momentum) is not mathematically conserved, instead a different pair of quantities (e.g., perhaps mass and energy) may be conserved, contrary to the original intent of the derivation.

Analysis of engineering equation forms

In the preceding sections, first two fundamental forms of the equations for unsteady flow in open channels are presented: the *basic* conservation form, generally defined as conservation-law form, and the corresponding *extended* conservation form. Then, more general forms, referred to as *expanded* forms, for use in engineering practice, which also belong to the quasi-linear first-order hyperbolic-type PDEs, are presented. However, the latter PDE forms exhibit a paradox, for they are shown either to conform mathematically to the afore-defined conservation forms or to transform into modified dissimilar forms. It is also demonstrated that different quantities are conserved for equation sets derived by using identical physical principles but different sets of dependent variables. That this paradox is inherent to the engineering PDE forms should signal caution to practicing engineers in the selection of equation forms for problem solving and in the choice of numerical models for hydraulic engineering applications. In view of this paradox, and in recognition of its engineering consequences, a brief analysis of engineering equation forms is offered in the following section. Six sets of unsteady open-channel flow equations resulting from different choices of dependent variables are shown to substantiate the preceding mathematical development. This equation analysis and presentation of engineering forms is subsequently followed by discussion of a set of numerical experiments that not only further enhance the mathematical development, but also demonstrate special properties of conservation forms in application to the analysis of flow discontinuities.

Derivation of unsteady flow equations

Equations for treating unsteady flow in a nonprismatic channel with lateral inflow, q, can be derived from the principles of mass and momentum conservation using cross-sectional area, A, and discharge, Q, as the dependent variables. The resulting equations [refer to Appendix I for derivation details] are:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - q = 0 \qquad (20)$$

$$\frac{\partial Q}{\partial t} + 2\left(\frac{Q}{A}\right)\frac{\partial Q}{\partial x} + \left[g\frac{A}{B} - \left(\frac{Q}{A}\right)^2\right]\frac{\partial A}{\partial x} + gAh_x^A - gA(S_b - S_f) - qu' = 0 \qquad (21)$$

in which q = lateral inflow per unit length of channel, u' = x-component of lateral-inflow velocity, g = gravitational acceleration, B = channel top width of flow, S_b = channel-bottom slope, S_f = friction slope, and

$h_x^A \equiv \left.\frac{\partial h}{\partial x}\right|_A$ = rate of change of flow depth, h, in the x-direction with cross-sectional area fixed.

Choice of dependent variables

Six sets of equations, representative of those used in engineering practice to treat unsteady flow in a nonprismatic channel having lateral inflow, have been derived similarly to Eqs. (20) and (21) from conservation of mass and momentum principles (Lai, 1986; Lai et al., 1998). In each set, the pair of dependent variables affects the final form of the equations, even though the same principles of physics are used in their derivations. For the dependent variables depth, h, water-surface elevation, Z, cross-sectional area, A, velocity, u, and discharge, Q, the equation sets, resulting from the six different combinations (A,u), (A,Q), (h,u), (h,Q), (Z,u), and (Z,Q), have the following matrix expressions in the form of Eq. (9), arranged in **U**, **M**, **N**, **E** order:

$$\begin{bmatrix} A \\ u \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ u & A \end{bmatrix}, \begin{bmatrix} u & A \\ u^2 + c^2 & 2uA \end{bmatrix}, \begin{bmatrix} -q \\ gAh_x^A - gA\tilde{S} - qu' \end{bmatrix};$$

$$\begin{bmatrix} A \\ Q \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ c^2 - \left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -q \\ gAh_x^A - gA\tilde{S} - qu' \end{bmatrix};$$

$$\begin{bmatrix} h \\ u \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ u & H \end{bmatrix}, \begin{bmatrix} u & H \\ u^2 + c^2 & 2uH \end{bmatrix}, \begin{bmatrix} \frac{u}{B}A_x^h - \frac{q}{B} \\ \frac{u^2}{B}A_x^h - c^2\tilde{S} - \frac{qu'}{B} \end{bmatrix};$$

$$\begin{bmatrix} h \\ Q \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{B} \\ Bc^2 - B\left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -\frac{q}{B} \\ -\left(\frac{Q}{A}\right)^2 A_x^h - gA\tilde{S} - qu' \end{bmatrix};$$

$$\begin{bmatrix} Z \\ u \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ u & H \end{bmatrix}, \begin{bmatrix} u & H \\ u^2 + c^2 & 2uH \end{bmatrix}, \begin{bmatrix} \frac{u}{B}A_x^h + uS_b - \frac{q}{B} \\ \frac{u^2}{B}A_x^h + u^2S_b + c^2S_f - \frac{qu'}{B} \end{bmatrix};$$

$$\begin{bmatrix} Z \\ Q \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{B} \\ Bc^2 - B\left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -\frac{q}{B} \\ -\left(\frac{Q}{A}\right)^2 A_x^h - B\left(\frac{Q}{A}\right)^2 S_b + gAS_f - qu' \end{bmatrix};$$

in which $H = A/B$, $c = \sqrt{gA/B}$, $\tilde{S} = (S_b - S_f)$, and $A_x^h \equiv \left.\frac{\partial A}{\partial x}\right|_h$ = rate of change of cross-sectional area in the x-direction with depth fixed.

Note that (A,Q), (h,Q) and (Z,Q) equation sets, referred to hereafter as the first group, have **M** = **I**, and thus, are directly in the form of Eq. (7) [or Eq. (13)]. Pre-multiplying each term in the (A,u), (h,u) and (Z,u) equation sets, hereafter called the second group, by the corresponding $\mathbf{M}^{-1}$ matrix, or subtracting the equation of continuity from that of motion in each set, will reduce these equation sets to the form of Eq. (7) [or Eq. (13)], as well.

Written in order of terms **U**, **A** (= **K**), and –**G** (= **B**), the six resultant equation sets are:

$$\begin{bmatrix} A \\ u \end{bmatrix}, \begin{bmatrix} u & A \\ \frac{c^2}{A} & u \end{bmatrix}, \begin{bmatrix} -q \\ -g\tilde{S} + \frac{q\upsilon}{A} + gh_x^A \end{bmatrix};$$

$$\begin{bmatrix} A \\ Q \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ c^2 - \left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -q \\ gAh_x^A - gA\tilde{S} - qu' \end{bmatrix};$$

$$\begin{bmatrix} h \\ u \end{bmatrix}, \begin{bmatrix} u & H \\ g & u \end{bmatrix}, \begin{bmatrix} \frac{u}{B}A_x^h - \frac{q}{B} \\ -g\tilde{S} + \frac{q\upsilon}{A} \end{bmatrix};$$

$$\begin{bmatrix} h \\ Q \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{B} \\ Bc^2 - B\left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -\frac{q}{B} \\ -\left(\frac{Q}{A}\right)^2 A_x^h - gA\tilde{S} - qu' \end{bmatrix};$$

$$\begin{bmatrix} Z \\ u \end{bmatrix}, \begin{bmatrix} u & H \\ g & u \end{bmatrix}, \begin{bmatrix} \frac{u}{B}A_x^h + uS_b - \frac{q}{B} \\ gS_f + \frac{q\upsilon}{A} \end{bmatrix};$$

$$\begin{bmatrix} Z \\ Q \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{B} \\ Bc^2 - B\left(\frac{Q}{A}\right)^2 & \frac{2Q}{A} \end{bmatrix}, \begin{bmatrix} -\frac{q}{B} \\ -\left(\frac{Q}{A}\right)^2 A_x^h - B\left(\frac{Q}{A}\right)^2 S_b + gAS_f - qu' \end{bmatrix};$$

in which $\upsilon = u - u'$.

Notice that pre-multiplication of the second group of equation sets by $\mathbf{M}^{-1}$ has changed the equation of motion in each equation set to an energy-form equation. This development clearly demonstrates the importance of the choice of dependent variables in determining the conservation form and properties of the resultant equation set.

Application and solution needs affecting choice of equation forms

Selection of an equation form consistent with the range and scope of prototype flows to be simulated is typically the first consideration in the development of an unsteady, open-channel flow model. Likewise, it is an equally important consideration for a model user seeking to adapt an existing model to a prototype flow condition. After the physical quantities to be conserved have been identified and an equation set selected, other aspects of equation form, such as the extent of nonhomogeneous terms needed and the type of boundary conditions required, must be considered.

Nonhomogeneous Terms: The extent of nonhomogeneous terms included in a given equation set largely determines the diversity and scope of prototype flow conditions able to be addressed and simulated. From inspection of the nonhomogeneous terms, **B** (or –**G**), in the six extended-form equation sets above, it is apparent that this two-component, column vector tends to exhibit considerable complexity. Clearly, the magnitude of the vector varies appreciably as each of the four nonhomogeneous quantities--lateral inflow, nonprismatic channel geometry, friction slope, and bed slope--included in the equation sets, above, increase or decrease in value. Moreover, by including other external forcing functions, e.g., surface wind stress, Coriolis effects, etc., the scope of the model can be expanded, but at the expense of an attendant increase in complexity of the nonhomogeneous term vector. Preliminary investigations of the degree to which the inclusion and magnitude of various nonhomogeneous terms affect model behavior and simulation results have been reported previously (Lai et al., 1987; Schaffranek and Lai, 1994). However, a more extensive investigation of the effects of the inclusion and varying magnitudes of these terms, with particular attention to their relation to conservation form, is desirable.

Boundary Conditions: The types of dependent variables used in the selected equation set often affect the types of boundary-value data needed. Thus, selection of the *(h,Q)* equation set, for example, dictates that direct input of boundary values consist of time sequences of depth or discharge data. In a practical sense, however, the ease of acquisition and cost of obtaining one type of boundary-value data vis-à-vis another (e.g., depth versus discharge data) are typical factors determining data availability. The response of a particular model to different types and combinations of boundary-value data is also a factor for consideration in selecting an equation set. The results of sensitivity analyses conducted to determine friction term response to different types and combinations of boundary-value data indicate that other factors influence this choice as well (Schaffranek and Lai, 1996).

Selection of different equation forms having different dependent variables will result in simulation models having different boundary-value data requirements, different levels of complexity of the nonhomogeneous terms, and different degrees of applicability for simulating unsteady, open-channel flows. However, it is particularly important to recognize that neither a change in the scope or extent of the nonhomogeneous terms, nor insightful choices of boundary-value data types will, in any way, affect the basic 'hyperbolicity' of an equation set or its 'characteristics' behavior, e.g., for the six equation sets examined, above, the conservation properties of each remains unaltered.

Numerical experiments

A set of numerical experiments was devised to test the validity of the prior theoretical development and to demonstrate quantitatively the conservation nature of various equation forms. These experiments, focused on application of conservation-form equations to a flow-discontinuity problem, were performed for both steady- and unsteady-flow conditions using a hypothetical channel, in which flow transitions from super-critical to sub-critical state.

Continuous and discontinuous flows

Solutions of the six conservation-form equation sets above should produce the same results for continuous flows. Expanding on this concept, Abbott (1979) points out that given the respective equations of mass, momentum and energy conservation for (continuous) unsteady, open-channel flow, any two of these equations should suffice for a unique solution. On the other hand, for a flow discontinuity, e.g., in which both super-critical and sub-critical flows occur, different solutions result from different equation sets. For a hydraulic jump, for instance, the total momentum before and after the jump is typically invariant. Thus, momentum is conserved through the jump, however energy is not. By means of a channel constriction it is equally possible to transition from super- to sub-critical flow (or the reverse) with full conservation of energy; momentum, however, is not conserved. In a flow discontinuity, by virtue of the fact that the corresponding momentum- and energy-related depths in the sub-critical regime, normally referred to as the conjugate and alternate depths, respectively, are *not* equal, it is possible to experimentally demonstrate whether a particular equation set conserves mass and momentum or mass and energy.

Design

For a flow discontinuity in an open channel, a model based upon any of the equation sets from the original conservation-form group, (A,Q), (h,Q), and (Z,Q), the first group, would be expected to simulate conjugate-depth change downstream of the transition because they conserve momentum. Conversely, a model based upon any of the three transformed equation sets, (A,u), (h,u), and (Z,u), the second group, would be expected to simulate alternate-depth change because they conserve energy.

Two equations sets, (h,Q) from the first group and (h,u) from the second group, were used to conduct the numerical experiments. When written in fully expanded form similarly to Eqs. (20) and (21), the continuity and momentum equations for (h,Q) are:

$$\frac{\partial h}{\partial t}+\frac{1}{B}\frac{\partial Q}{\partial x}-\frac{q}{B}=0 \tag{22}$$

$$\frac{\partial Q}{\partial t}+2\left(\frac{Q}{A}\right)\frac{\partial Q}{\partial x}+B\left[c^2-\left(\frac{Q}{A}\right)^2\right]\frac{\partial h}{\partial x}-\left(\frac{Q}{A}\right)^2 A_x^h -gA\tilde{S}-qu'=0 \tag{23}$$

respectively, and for (h,u) the equations are:

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+H\frac{\partial u}{\partial x}+\frac{u}{B}A_x^h-\frac{q}{B}=0 \tag{24}$$

$$u\frac{\partial h}{\partial t}+H\frac{\partial u}{\partial t}+(u^2+c^2)\frac{\partial h}{\partial x}+2uH\frac{\partial u}{\partial x}+\frac{u^2}{B}A_x^h -c^2\tilde{S}-\frac{qu'}{B}=0. \tag{25}$$

A numerical model using an implicit, four-point, single/double-sweep technique was developed to solve the equations. To eliminate the potential for differences resulting from the numerical process, this model was specifically designed to solve either the (h,Q) equation set (Eqs. 22 and 23) or the (h,u) equation set (Eqs. 24 and 25).

Experimental conditions

A hypothetical channel comprised of three sub-reaches, as shown in figure 1, is used to conduct the flow experiments. Super-critical flow always occurs in the first (most upstream) sub-reach; sub-critical flow always occurs in the third (most downstream) sub-reach. A surface discontinuity is designed to occur in the second (middle) sub-reach in which super-critical inflow transitions to sub-critical outflow before entering the third sub-reach. The flow transition occurs either as a hydraulic jump having the resultant conjugate depth, or as a constrictive-flow transition having the appropriate alternate depth. Strictly speaking, the rapid change in water level induced by application of the hydraulic-jump equation is a discontinuous flow whereas that induced by application of the Bernoulli-type energy equation is not. It should be noted, however, that because the numerical simulations are conducted on a discrete grid that is larger in length than the transition limits, the experimental effects and subsequent analyses based on them remain valid and appropriate.

At the upstream end of the channel, two boundary conditions are required because of the prevailing super-critical flow conditions. Depths and either discharges or velocities are specified depending on which equation set, (h,Q) or (h,u), is being solved. In these experiments, discharge is specified, therefore, uniform depth, h, and corresponding velocity, u, for steady flow can be determined. At the downstream end of the channel, only one boundary condition is needed due to the prevailing sub-critical flow in the third sub-reach. Depths are specified because they are shared by both equation sets. Constant depth is used for the steady-flow experiments and time-varying critical depth for the unsteady-flow experiments. Flow enters the third sub-reach with either conjugate or alternate depths, i.e., the internal boundary condition, depending on the (designed) discontinuity condition.

The hypothetical channel is of uniform rectangular cross section having a width, B, of 6 m and a total length, L, of 2100 m. The channel is subdivided into three sub-reaches of lengths $L_1 = 1000$ m, $L_2 = 100$ m, and $L_3 = 1000$ m, respectively. Sub-reaches L_1 and L_3 are each further subdivided into $\Delta x = 100$-m segments, as illustrated in figure 1. The channel-bottom gradients are $S_1 = 0.0064$, $S_2 = 0.0$, and $S_3 = 0.0064$; and the frictional-resistance coefficients are $\eta_1 = 0.01200$, $\eta_2 =$ unspecified, $\eta_3^c = 0.03416$, and $\eta_3^a = 0.03883$, where the superscripts, c, and, a, denote conjugate and alternate depth associations, respectively.

Two conditions are stipulated to occur within the middle sub-reach, L_2, either a hydraulic jump or a constricted transition, dependent on the particular numerical experiment being conducted.

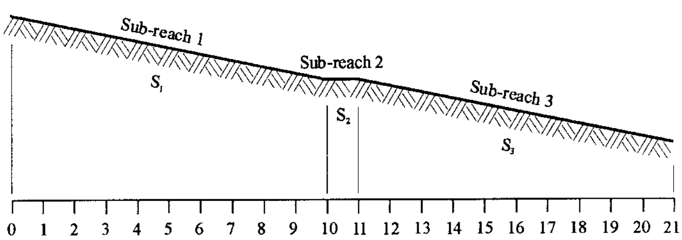

Fig. 1. Profile of hypothetical, uniform rectangular channel used for flow experiments.

A hydraulic jump is initiated by appropriate design of the channel-profile configuration whereas a constricted transition is initiated by reducing the channel width. For the conditions of these numerical experiments, a change in the channel width creating a constriction, $B_{min} = 4.868$ m, in the middle sub-reach is sufficient to initiate a flow transition without causing a choke (Henderson, 1966). Both of these flow-transition conditions are of micro-(sub-grid-size)scale by comparison with the macro-(grid-size)scale sub-reach within which they occur. The exact nature and behavior of the micro-scalar mechanisms are inconsequential to the numerical experiments as long as conjugate and alternate depths are induced by the respective flow-transition conditions within the middle sub-reach, L_2.

A discharge, Q, of 12 m^3s^{-1} with corresponding uniform-flow depth, h_0, is used as a boundary condition at the channel entrance to conduct the steady-flow simulations and as the base-flow condition for the unsteady-flow simulations. Unsteady-flow conditions are generated by imposing a linear increase in depth from h_0 to the peak depth, h_p, in one hour, followed by a linear return to base-flow conditions (i.e., to h_0) in two hours. A time-varying sequence of discharges or velocities (again, depending on which equation set is being solved) is used to satisfy the second boundary condition at the channel entrance. By holding the Froude number invariant with time, the required sequence of discharges or velocities is determinable. (At peak depth, $h_p = 1.1398$ m, $Q_p = 39.22$ m^3s^{-1} and $v_p = 5.735$ ms^{-1}.) The full set of initial conditions plus internal and external boundary conditions used to conduct the numerical experiments is presented in Appendix II.

Simulation results

Figures 2 and 3 show simulated, steady-flow depths and discharges for assumptions of energy and momentum conservation with respect to the (h,u) or (h,Q) equation sets. Corresponding results from the unsteady-flow simulations are plotted in figures 4 and 5.

Steady- and unsteady-flow depth and discharge hydrographs in figures 2-5 clearly illustrate the compatibility between the (h,Q) equation set and a momentum conservation assumption, and between the (h,u) set and an energy conservation assumption. Incompatibilities between equation sets and the effects of the imposition of inconsistent internal boundary conditions at the flow transition on simulation results are readily apparent also.

Discussion of results

Simulated steady-flow depths and discharges from the model based on the (h,u) equation set clearly match the energy-conserving alternate depths and discharges as can be seen in figures 2a and 3a. Similar findings are evident for simulated unsteady-flow depths and discharges, which are in agreement with energy conservation principles as illustrated in figures 4a and 5a. On the other hand, for a model based on the (h,Q) equation set, simulated results show excellent agreement with momentum conserving conjugate depths and discharges, as can be seen in figures 2d and 3d for steady flow and in figures 4d and 5d for unsteady flow.

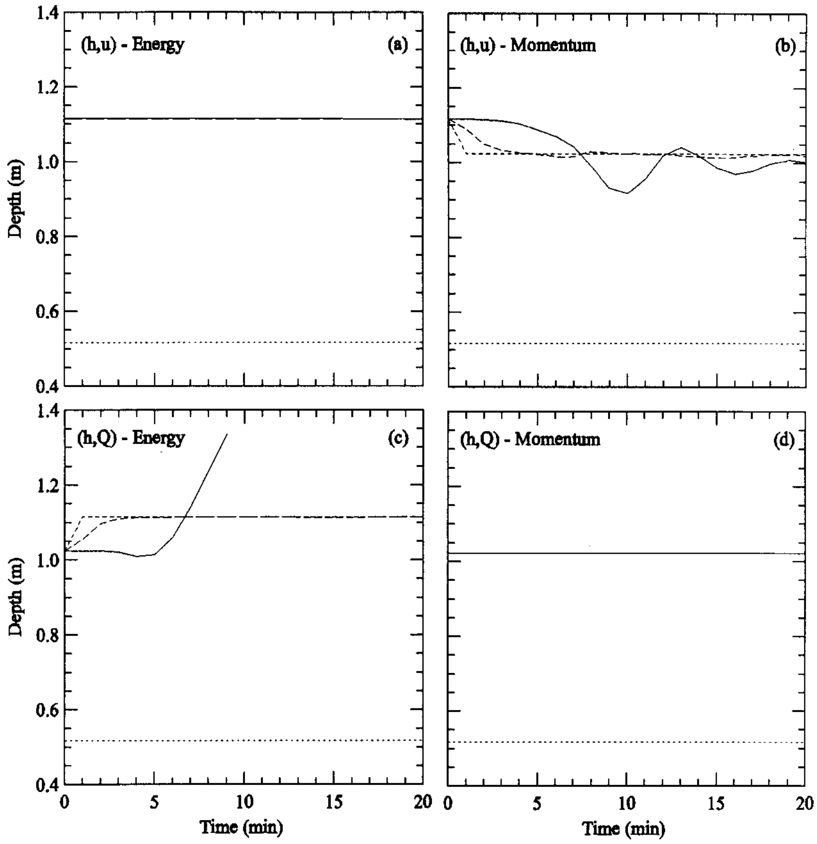

Fig. 2. Steady-flow depths computed at 800 (dot), 1100 (short dash), 1300 (long dash), and 2000 (solid) meters from upstream boundary, using depth (h), velocity (u), and discharge (Q) dependent-variable combinations with alternate or conjugate depths specified at the internal flow transition.

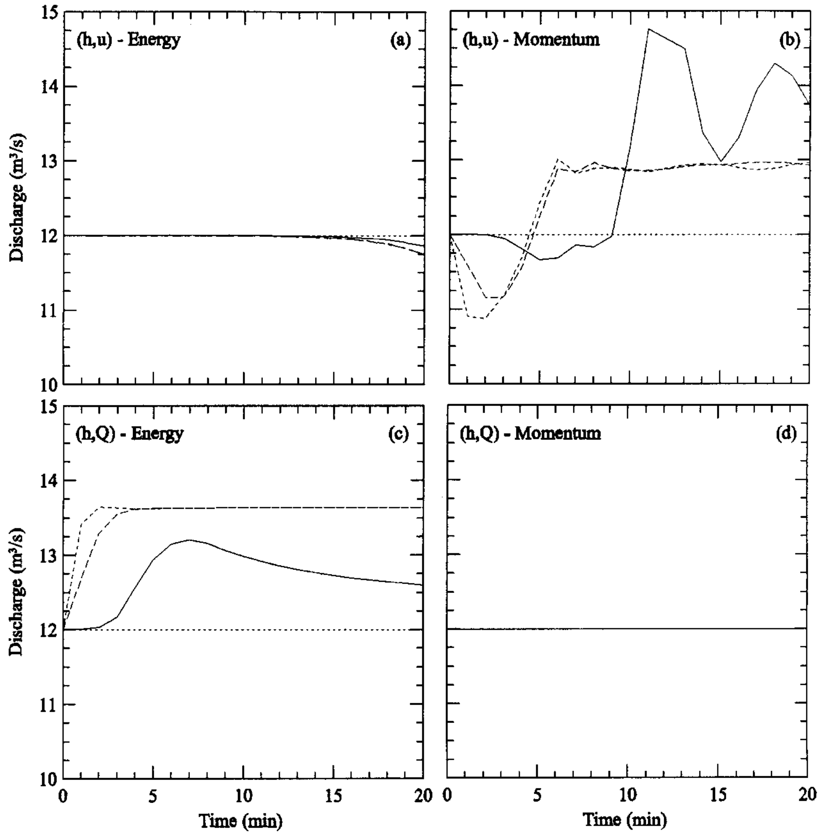

Fig. 3. Steady-flow discharges computed at 800 (dot), 1100 (short dash), 1300 (long dash), and 2000 (solid) meters from upstream boundary, using depth (h), velocity (u), and discharge (Q) dependent-variable combinations with alternate or conjugate depths specified at the internal flow transition.

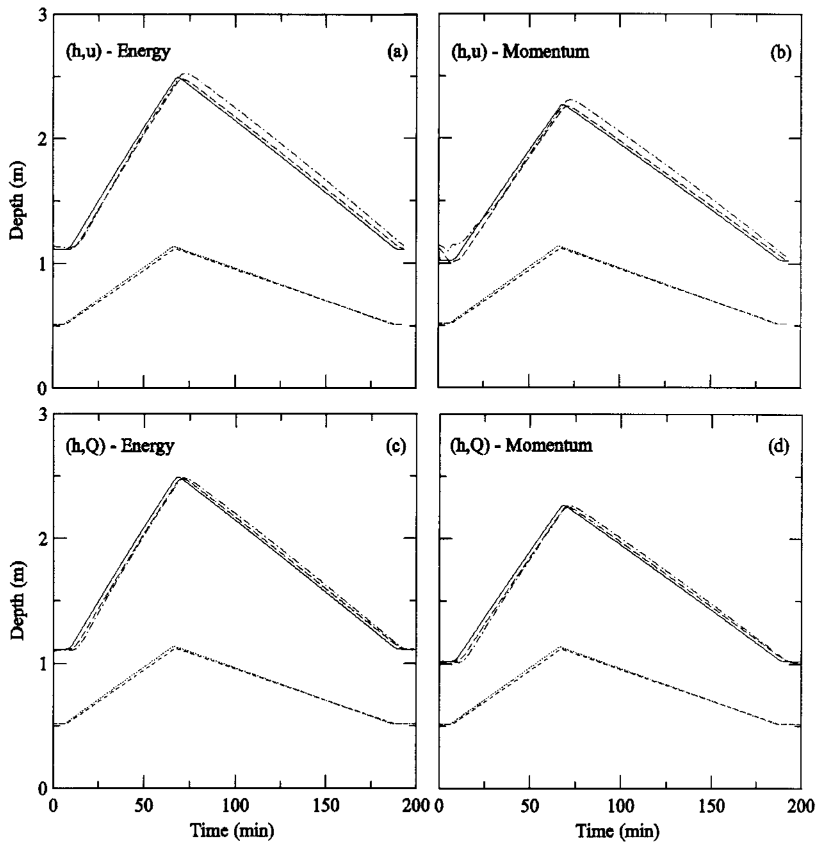

Fig. 4. Unsteady-flow depths computed at 0 (dot), 500 (short dash), 1100 (solid), 1600 (long dash), and 1900 (dash dot) meters from upstream boundary, using depth (h), velocity (u), and discharge (Q) dependent-variable combinations with alternate or conjugate depths specified at the internal flow transition.

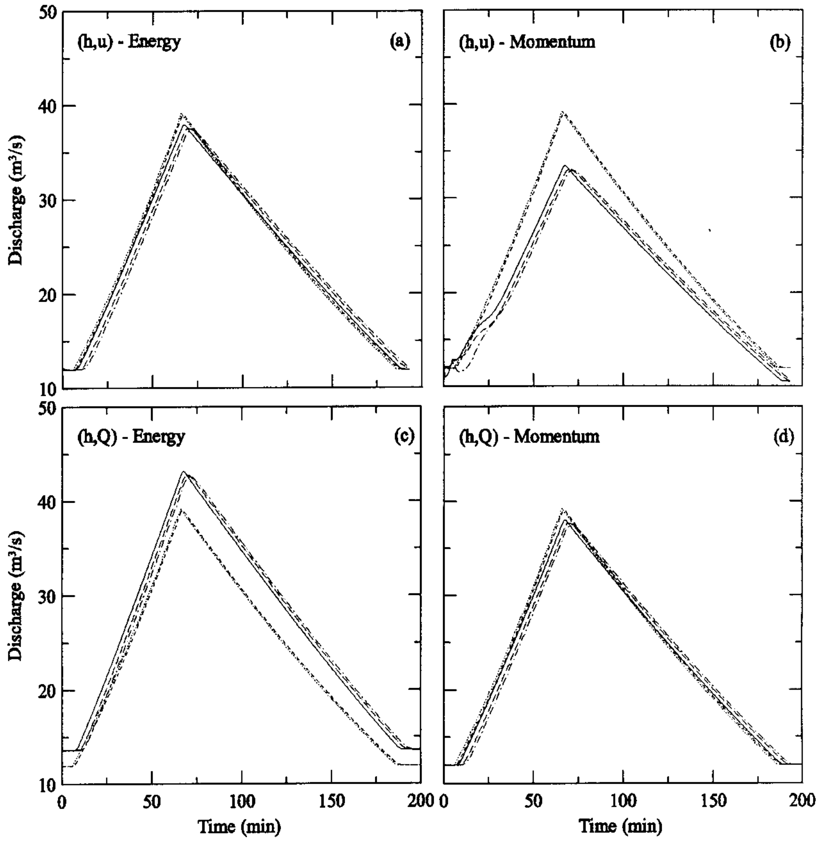

Fig. 5. Unsteady-flow discharges computed at 0 (dot), 500 (short dash), 1100 (solid), 1600 (long dash), and 1900 (dash dot) meters from upstream boundary, using depth (h), velocity (u), and discharge (Q) dependent-variable combinations with alternate or conjugate depths specified at the internal flow transition.

However, numerical results from the model using an inappropriate conservation assumption, e.g., an (h,Q)-based model with an assumption of mass/energy conservation, or an (h,u)-based model with an assumption of mass/momentum conservation, evidence various degrees of instability or anomalies as can be seen in figures 2b, 2c, 3b, and 3c, for steady flows and in figures 4b, 4c, 5b, and 5c, for unsteady flows. Fluctuations or erroneous simulated depths and discharges evidence that quantities are not being conserved. For the unsteady-flow experiments, although simulated results are shown to be deceptively smooth, the sudden changes in discharges after the discontinuity, depicted in figures 5b and 5c, are clearly violations of the conservation principles and the corresponding depths in figures 4b and 4c are wrong because the imposed internal boundary condition is inconsistent with the governing flow-equation set.

The divergence of discharges manifested in the simulation results at the end of the steady-flow experiment presented in figure 3a, illustrates the difficulties that can arise through use of a comprehensive unsteady-flow model to simulate steady-flow conditions for an extended period of time. In fact, this difficulty has been pointed out by LeVeque (1992) and identified as a special field of study in its own right, which is well beyond the scope of the present effort. That this numerical artifact occurs does not alter or diminish the significance of the use of the creditable results to analyze alternate conservation-form equations.

To confirm the obtained simulation results and to investigate the effect of the channel gradient on findings, the numerical experiments were repeated for a channel having a different gradient but otherwise similar attributes. For this set of experiments, the gradients in the hypothetical channel sub-reaches were reduced to $S_1 = 0.0036$, $S_2 = 0.0$, and $S_3 = 0.0036$, respectively. Again, supercritical, uniform flow prevailed at all times throughout the first sub-reach; sub-critical, uniform flow prevailed at all times throughout the third sub-reach; and the flow transition was confined to the second (middle) sub-reach. The same steady-flow discharge of 12 m^3s^{-1} (with an appropriate corresponding initial depth, h_0) and unsteady-flow hydrograph were used for this set of experiments. Appropriate boundary values were specified at the channel extremities and either conjugate or alternate depths were imposed as internal boundary conditions at the entrance to the third sub-reach. Simulated steady- and unsteady-flow depth and discharge hydrographs were of similar character and entirely consistent with those obtained from the original experiments as presented in figures 2-5. For this reason these results are not presented herein.

In summary, the numerical results of these experiments indicate that:

- the (h,u) equation set is consistent with alternate flow depth and conservation of energy,
- the (h,Q) equation set is consistent with conjugate flow depth and conservation of momentum,
- the numerical behavior of both equation sets is consistent for steady and unsteady flows,
- for inappropriate (inconsistent) choices of transition depths and

dependent-variable combinations, simulated quantities are *not* conserved,

- computed unsteady-flow discharge hydrographs, determined using the (h,u) and (h,Q) equation sets with the appropriate alternate and conjugate depths, respectively, are nearly identical,
- computed unsteady-flow discharge hydrographs, determined using the (h,u) and (h,Q) equation sets with inappropriate conjugate and alternate depths, respectively, although seemingly stable and reasonably smooth, are observed to be erroneous upon close inspection, and
- the findings for both steady and unsteady flows in channels having different gradients concur.

Conclusions

A study of the conservation-law form of the unsteady open-channel flow equations has been conducted and has provided insight into the nature and importance of the conservation concept to the general field of numerical simulation modeling. Subsequent to an initial theoretical analysis of the meaning and significance of conservation-law form, a series of numerical experiments was conducted to demonstrate its relevance and underscore its implications for both developers and users of numerical flow simulation models in the engineering profession. The conservation properties of six sets of the unsteady, open-channel flow equations, formulated using different combinations of five possible dependent variables (discharge, velocity, depth, water-surface elevation, and cross-sectional area) were theoretically investigated. In addition, the conservation properties of two of these six equation sets were numerically demonstrated and contrasted. The numerical results confirm findings derived from the theoretical development. In particular, the theoretical analysis and numerical experiments clearly demonstrate that unsteady-flow equation sets using discharge as one of the dependent variables (the original group of conservation-form equations) conserve mass and momentum; whereas, those equation sets using velocity as a dependent variable (the transformed group) conserve mass and energy. Experimental results further confirm analytical findings that the choice of dependent variables determines whether or not a particular equation set conforms to conservation-law form and, moreover, what quantities are conserved.

As a consequence of this study, the following conclusions and summary remarks are appropriate:

1. The unsteady-flow equations can conform to the conservation law in *pure* or *basic* form in the absence of nonhomogeneous (source/sink) terms or in *extended* form if such terms are included.
2. An equation set that is *not* originally in conservation form, either *pure* or *extended,* can be converted to a conservation form, referred to as *reduced* form. However, by conversion, the type of equation is altered, thus, the quantities for which conservation was intended are no longer conserved, instead other quantities are conserved.
3. The pair of dependent variables used to derive unsteady-flow equations is the most important factor determining whether or not the equation set will be in conservation form.
4. The dependent variables used to derive unsteady-flow equations also affect the form and number of terms in the extended part of the equations consisting of the nonhomogeneous terms.
5. Analytical and numerical findings indicate that a model based on an equation set from the original group of conservation-form equations is appropriate to address flow transitions/discontinuities in which momentum is conserved, whereas, a model based on an equation set from the transformed group is appropriate to address problems in which energy is conserved. Both groups are appropriate for modeling continuous flows.
6. A model solving the equations of mass and momentum conservation is appropriate to address flow problems involving a surface discontinuity caused by a hydraulic jump.
7. A model solving the equations of mass and energy conservation (that may be *reduced* from their original form) is appropriate to address flow problems involving a surface discontinuity caused by a channel constriction.

A recommendation for future work is that other factors affecting equation forms, such as nonhomogeneous term selection and boundary condition treatment, should be more extensively investigated. Last but not least, numerical methods for solving the unsteady-flow equations consistent with the findings regarding conservation form reported herein should be thoroughly investigated.

Appendix I: Derivation of unsteady-flow equations for nonprismatic channels.

For a control volume of fixed channel length, δx, between longitudinal distances x_1 and x_2, referenced to an orthogonal Cartesian coordinate system, the following equations can be written:

Eq. of Mass Conservation

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}(\rho A)+\frac{\partial}{\partial x}(\rho Q)-\rho q\right]dx\,dt=0 \tag{26}$$

Eq. of Momentum Conservation

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}(\rho Au)+\frac{\partial}{\partial x}(\rho Qu)-\rho qu'\right]dx\,dt=\\&-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}(\gamma A\bar{h})dx\,dt+\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial x}(\gamma A\bar{h})\mid_h dx\,dt\\&-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\gamma A\frac{\partial Z_b}{\partial x}dx\,dt-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\gamma AS_f dx\,dt\end{aligned} \tag{27}$$

in which ρ = fluid density, γ = specific weight of fluid, $u = Q/A$ = flow velocity, q = lateral inflow per unit length, u'= x-component of lateral-inflow velocity, g = gravitational acceleration, $\bar{h}$ = depth of center of gravity of the cross section, $\gamma A\bar{h}$ = hydrostatic pressure, $\gamma\left(A\bar{h}\right)_x^h$ = hydrostatic pressure due to variation of channel width with h fixed, S_f = friction slope, and Z_b = channel-bottom elevation.

For most open-channel flow, ρ can be considered a constant (i.e., the fluid is incompressible and of homogeneous density). Also, from the functional relationships, $A = A[h(x,t),x]$; $h = h[A(x,t), x]$; $(A\bar{h}) = (A\bar{h})[h(x,t),x]$; etc.,

$$\frac{\partial A}{\partial x} = B\frac{\partial h}{\partial x} + A_x^h, \quad \frac{\partial A}{\partial t} = B\frac{\partial h}{\partial t};$$
$$\frac{\partial h}{\partial x} = \frac{1}{B}\frac{\partial A}{\partial x} + h_x^A, \quad \frac{\partial h}{\partial t} = \frac{1}{B}\frac{\partial A}{\partial t};$$
$$\frac{\partial}{\partial x}(A\bar{h}) = A\frac{\partial h}{\partial x} + (A\bar{h})_x^h, \quad \frac{\partial}{\partial t}(A\bar{h}) = A\frac{\partial h}{\partial t};$$

and so forth, in which B = channel top width and $A_x^h \equiv \left.\frac{\partial A}{\partial x}\right|_h$, etc. Equations (26) and (27) then can be rewritten as:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - q\right]dx\,dt = 0 \tag{28}$$

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) - qu'\right]dx\,dt = -g\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[A\frac{\partial h}{\partial x} - A(S_b - S_f)\right]dx\,dt \tag{29}$$

in which $S_b = -\frac{\partial Z_b}{\partial x}$ = channel-bottom slope. If it is assumed that the dependent variables are continuous and differentiable, Eq. (28) and Eq. (29) must hold for any infinitesimal volume in the (x,t) domain, and the differential form of the unsteady open-channel flow equations can be written:

Eq. of Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q \tag{30}$$

Eq. of Motion:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = -gA\frac{\partial h}{\partial x} + gAS_b - gAS_f + qu' \tag{31}$$

Inspection of Eqs. (30) and (31) reveals that the nonhomogeneous terms on the right-hand sides are those introduced to facilitate the practical utility of the equation set. Moreover, these equations belong to the category of Eqs. (6), (7), and (8). However, if the nonhomogeneous terms were all zero, Eqs. (30) and (31) would belong to the form of Eqs. (1), (2), and (4). To facilitate examination of the effects of the choice of dependent variables, Eqs. (30) and (31) are rewritten in the form of Eqs. (20) and (21).

Appendix II: Initial, internal-junction, and boundary conditions for numerical experiments.

A. Steady-flow case

Conjugate depth discontinuity:

$h(x_j, t_0)_{j=0,\dots,10} = 0.51754$ m, $h(x_j, t_0)_{j=11,\dots,21} = 1.02296$ m; and
$Q(x_j, t_0)_{j=0,\dots,21} = 12.00000$ m^3/s, or
$u(x_j, t_0)_{j=0,\dots,10} = 3.86444$ m/s, $u(x_j, t_0)_{j=11,\dots,21} = 1.95512$ m/s;
$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s,
$h(x_{21}, t_n) = 1.02296$ m, $t_0 < t_n$
A set of continuity and hydraulic-jump equations is imposed at L_2.

Alternate depth discontinuity:

$h(x_j, t_0)_{j=0,\dots,10} = 0.51754$ m, $h(x_j, t_0)_{j=11,\dots,21} = 1.11459$ m; and
$Q(x_j, t_0)_{j=0,\dots,21} = 12.00000$ m^3/s, or
$u(x_j, t_0)_{j=0,\dots,10} = 3.86444$ m/s, $u(x_j, t_0)_{j=11,\dots,21} = 1.79438$ m/s;
$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s,
$h(x_{21}, t_n) = 1.11459$ m, $t_0 < t_n$
A set of continuity and Bernoulli equations is imposed at L_2.

B. Unsteady-flow case

Conjugate depth discontinuity:

$h(x_j, t_0)_{j=0,\dots,10} = 0.51754$ m, $h(x_j, t_0)_{j=11,\dots,21} = 1.02296$ m; and
$Q(x_j, t_0)_{j=0,\dots,21} = 12.00000$ m^3/s, or
$u(x_j, t_0)_{j=0,\dots,10} = 3.86444$ m/s, $u(x_j, t_0)_{j=11,\dots,21} = 1.95512$ m/s;
$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s, $t_0 < t_n \le t_1$,
$h(x_0, t_n) = 0.0001728(t_n - t_1) + 0.51754$ m,

$u(x_0, t_n) = \mathscr{F}_0\sqrt{gh(x_0, t_n)}$ m/s, $t_1 < t_n \le t_2$,

$h(x_0, t_n) = -0.00008643(t_n - t_2) + 1.1398$ m,
$u(x_0, t_n) = \mathscr{F}_0\sqrt{gh(x_0, t_n)}$ m/s, $t_2 < t_n \le t_3$,

$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s, $t_3 < t_n$,
where $t_0 = 0$ s, $t_1 = 360$ s, $t_2 = 3960$ s, $t_3 = 11160$ s, and

$\mathscr{F}_0 = u(x_0, t_0)/\sqrt{gh(x_0, t_0)}$;

$h(x_{21}, t_n) = \left(\frac{[Q(x_{21}, t_n)/B]^2}{g}\right)^{1/3}$ m/s or $h(x_{21}, t_n) = \frac{[u(x_{21}, t_n)]^2}{g}, t_0 < t_n$.
A set of continuity and hydraulic-jump equations is imposed at L_2.

Alternate depth discontinuity:

$h(x_j, t_0)_{j=0,\dots,10} = 0.51754$ m, $h(x_j, t_0)_{j=11,\dots,21} = 1.11459$ m; and
$Q(x_j, t_0)_{j=0,\dots,21} = 12.00000$ m^3/s, or
$u(x_j, t_0)_{j=0,\dots,10} = 3.86444$ m/s, $u(x_j, t_0)_{j=11,\dots,21} = 1.79438$ m/s;
$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s, $t_0 < t_n \le t_1$,
$h(x_0, t_n) = 0.0001728(t_n - t_1) + 0.51754$ m,

$u(x_0, t_n) = \mathscr{F}_0\sqrt{gh(x_0, t_n)}$ m/s, $t_1 < t_n \le t_2$,

$h(x_0, t_n) = -0.00008643(t_n - t_2) + 1.1398$ m,

$u(x_0, t_n) = \mathscr{F}_0\sqrt{gh(x_0, t_n)}$ m/s, $t_2 < t_n \le t_3$,

$h(x_0, t_n) = 0.51754$ m, $u(x_0, t_n) = 3.86444$ m/s, $t_3 < t_n$,
where $t_0 = 0$ s, $t_1 = 360$ s, $t_2 = 3960$ s, $t_3 = 11160$ s, and

$\mathscr{F}_0 = u(x_0, t_0)/\sqrt{gh(x_0, t_0)}$;

$$h(x_{21}, t_n) = \left(\frac{[Q(x_{21}, t_n)/B]^2}{g}\right)^{1/3} \text{ m/s or } h(x_{21}, t_n) = \frac{[u(x_{21}, t_n)]^2}{g}, t_0 < t_n.$$

A set of continuity and Bernoulli equations is imposed at L_2.

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Notations

A	cross-sectional flow area
B	channel top width of flow
c	wave celerity (= $\sqrt{gA/B}$)
F_f	frictional-resistance force
g	gravitational acceleration
h	flow depth
$\bar{h}$	depth to center of gravity of cross section from water surface
H	hydraulic depth $(=A/B)$
L	channel length
n	Manning frictional-resistance coefficient
q	lateral flow discharge per unit channel length
Q	flow discharge
S_b	channel-bottom slope
S_f	friction slope
$\tilde{S}$	$S_b - S_f$
t	time
u	cross-sectional flow velocity
u'	x-component of lateral flow velocity
x	longitudinal distance along the channel
Δx	control volume length
Z	water-surface elevation
Z_b	channel-bottom elevation
γ	specific weight of fluid
η	resistance coefficient in unsteady flow (similar to Manning n)
λ	a characteristic
ρ	fluid density
υ	u-u'